Upper Bounds for the Tail of the Compound Negative Binomial Distribution

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Declaration

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Signature: ___________________ Date: ______________

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Abstract

Willmot & Lin (1997b, 2001) proposed an upper bound for the tail probabilities of the compound negative binomial distribution in the form of a (possibly degenerate) mixture of tails of Erlang distributions. Though sharp in the case of compound Pascal-exponential distributions, they provide no indication of the tightness of their upper bound in other cases. This essay extends their work, providing numerical illustrations for a wider range of negative binomial parameters and for degenerate (§2.1), exponential (§2.2), gamma (§2.3) and mixtures of exponential (§2.4) loss distributions, as well as investigating the importance of finding an optimal upper bound for practical purposes of reserve calculation (§3.1) and stop-loss reinsurance (§3.2)

Approximate word count: 9,200 words

Contents

Chapter 1

Introduction & Preliminaries

Compound distributions (also known as random sums) are used in modelling aggregate claims or losses of an insurance portfolio. They are the distributions arising from the sum of N independent, identically distributed random variables, Y_i, where N itself is a random variable. The total claims is therefore

$$S = Y_1 + Y_2 + \ldots + Y_N,$$

with $S = 0$ if $N = 0$

Compound distributions are classified by their counting distribution, N, the number of claims arising from the portfolio, while the associated individual random variable, Y_i, follows a loss distribution, representing individual loss amounts.

When the counting distribution follows a negative binomial distribution, we say the aggregate claims follow a compound negative binomial model. These models arise in insurance applications, such as motor vehicle and storm insurance. Suppose that the number of accidents or storms follows a Poisson distribution with the unknown Poisson parameter characterising the risk. If we assume the risk parameter follows a Gamma distribution, then accidents arising from the group of policyholders is a negative binomial random variable (Lemaire, 1995).

Also, in the case of multiple claims arising from one accident, if the number of accidents follows a Poisson distribution and the number of claims arising from each accident follows a Logarithmic distribution then the total number of accident claims again follows a negative binomial distribution (Panjer and Willmot, 1992)

Various insurance applications, such as reserve calculations and stop-loss reinsurance premiums are closely related to the tail probabilities of such aggregate distributions. However explicit expressions for the tail of a compound negative binomial distribution are only available for special cases. Hence there is great interest in determining estimates and bounds for these quantities.

There have been a large number of research papers published on exact calculation of compound distributions. These include matrix methods by Hesselager (1994) and Wang and Sobrero (1994) and recursive methods by Waldmann (1996), though the most widely used approach follows that of Panjer (1981) and Sundt and Jewell (1982), where recursive calculation is possible for a special class of counting distributions. In this paper, we employ Panjer's method to calculate the 'true' aggregate distribution under our compound negative binomial model.

Panjer's method however requires extensive calculations, and requires discrete loss distributions. Numerous researchers such as Lundberg (1940), Embrechts, Maejima and Teugels (1984) and Cai and Garrido (2002) have investigated approximations and bounds of compound distributions, though in this paper we focus on an upper bound to the compound negative binomial distribution proposed by Willmot and Lin (1997b). The objective of this essay is to determine when Willmot and Lin's upper bound is tight enough for practical purposes.

1.1. Panjer's method

Panjer's (1981) recursion requires the counting distribution to be of the form

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1}.$$

The negative binomial distribution has probability function

$$\begin{aligned} p_n &= \binom{\alpha + n - 1}{n} (1-\phi)^\alpha \phi^n, \quad n = 0,1,2,\ldots \\ &= \phi\left(1 + \frac{\alpha - 1}{n}\right) p_{n-1}, \end{aligned}$$

with $\alpha > 1$ and $0 < \phi < 1$. Thus, the negative binomial counting distribution is of the required form, with $a = \phi$ and $b = \phi(\alpha - 1)$. Panjer's recursion requires discrete loss distributions, so we divide each unit interval into b subdivisions and define

$$\hat{F}\left(\tfrac{x}{b}\right) = F\left(\tfrac{x}{b}\right), \quad x = 1,2,\ldots$$

In this manner, the discretised loss distribution never overestimates the true distribution and as $b \to \infty$, approaches the continuous loss distribution from below as we observe in Figure 1.

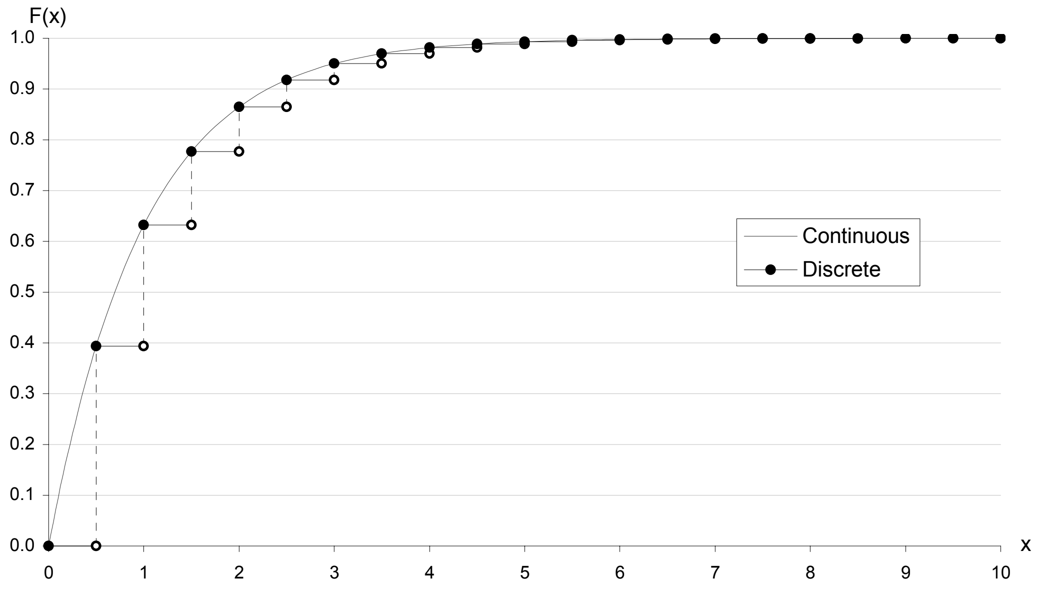


Figure 1: Discretised loss distribution (b = 2)

Similarly, aggregate claims under the discrete distribution are underestimated.

$$\hat{G}(x) = \sum_{n=1}^{\infty} p_n \hat{F}^{*n}(x) < \sum_{n=1}^{\infty} p_n F^{*n}(x) = G(x), \quad x \geq 0,$$

so Panjer's method always overestimates the tail distribution, $\bar{G}(x) = 1 - \hat{G}(x)$.

Our recursion for the compound negative binomial distribution is then

$$g_x = \sum_{y=1}^{xb} \phi \left(1 + \frac{(\alpha - 1) y}{x} \right) \hat{f}_{\frac{y}{b}} g_{x - \frac{y}{b}}, \quad x = \tfrac{1}{b}, \tfrac{2}{b}, \ldots$$

$$g_0 = (1 - \phi)^{\alpha}.$$

Panjer and Wang (1993) show that the recursion formula is stable for all $x \geq 0$. Thus the above algorithm requires no additional handling of rounding errors and the precision of modern computers is sufficient to obtain meaningful results.

For our purposes, we only require values up to the 99.5th percentile, so we may cease Panjer's recursion at $x = x_{\max}$, the value at which the cumulative tally first exceeds 0.995. We form the cumulative density function $G(x)$ by accumulating values of g_x, and then calculate percentiles up to and including $G(x_{\max})$ by linear interpolation of adjacent data points (for continuous loss distributions), or by the smallest data point of equal or greater value (for discrete loss distributions).

Our value of b is chosen manually by subdividing the los distribution and repeating the above procedure until any extra division leads to less than a 1% improvement in the 99.5th percentile.

Usually, x_{max} is bounded (i.e. $x_{max} \leq 5000$), however this may force b to be lowered, otherwise early termination may result in undefined percentiles. Such a limitation would prevent the aggregate distribution converging, and reduce the accuracy of any results, For this reason, x_{max} remained unbounded.

Panjer's recursion method requires $x(x + 1)/2$ summation terms to obtain a value for $g_{x/b}$. Thus for a given loss distribution, finer subdivisions result in a drastic increase in the amount of computation required to calculate g_x. This places a severe practical limitation on the accuracy and extent of Panjer's method.

For this investigation, Panjer's method was implemented in VBA under Excel, on a Pentium 466Mhz machine. The most extensive example required 12,600 iterations and completed in 26 minutes 47 seconds.

1.2. Willmot and Lin's method

An alternative that avoids extension calculations is Willmot and Lin's (1997b, 2001) method. They propose upper bounds for the tail of the compound negative binomial distribution as a mixture of tails of Erlang distributions. For computational simplicity we use Willmot and Lin's (2001, §7.4.10) result.

$$\overline{G}(x) \le e^{-\kappa x} \sum_{i=0}^{m-1} \overline{R}_i \frac{(\kappa x)^i}{i!}, \quad x \ge 0,$$

where κ > 0 satisfies

$$\int_0^{\infty} e^{\kappa y} dF(y) = \frac{1}{\phi},$$

and

$$\overline{R}_i = \sum_{j=i+1}^{m} r_j, \quad i = 0, 1, 2, \ldots, m-1,$$

with $\left\{a_j\right\}_{j=1}^{m}$ satisfying $0 \le \alpha_j \le 1$ and $\sum_{j=1}^{m} \alpha_j = \alpha$, while $\left\{r_j\right\}_{j=1}^{m}$ satisfy

$$\sum_{j=0}^{m} r_j z^j = \prod_{i=1}^{m} \left(1 - C_{\alpha_i} + C_{\alpha_i} z\right)$$

$$C_{\alpha} = \frac{\theta}{\phi}\left\{1 - (1-\phi)^{\alpha}\right\}$$

$$\frac{1}{\theta} = \inf_{z \ge 0, \overline{F}(z) > 0} \int_z^{\infty} e^{\kappa y} dF(y) \Big/ e^{\kappa z} \overline{F}(z).$$

Lundberg (1903) was the first person to extensively investigate compound models in non-life insurance. In their broadest interpretation, his approximations yield exponential inequalities and first order asymptotic expansions for compound distributions with applications tin ruin estimation and total claim approximations.

Though Willmot and Lin establish Lundberg's original approximations for short-tailed distributions they also derive thicker tailed bounds than those of Lundberg's exponential case for long-tailed distributions (such as Pareto).

Willmot and Lin's method requires m summation terms, where m is the number of 'partitions' and is similar in magnitide to the negative binomial parameter α. This represents a reduction in computation time over Panjer's method, effectively determining a constant time algorithm that can calculate upper bouns for tail probabilities or upper percentiles independent of x. In determining upper percentiles, we use a Newton-Raphson approach

$$x_{i+1} = x_i + \frac{\bar{G}(x) - \bar{G}(x_i)}{\bar{G}'(x_i)},$$

where $\bar{G}(x)$ is the desired tail probability. Although there is a closed form for the derivative, for ease of implementation we use

$$\bar{G}'(x_i) \approx \frac{G(x_i + \delta) - G(x_i)}{\delta},$$

for $\delta = 0.0001$. This approach yields the desired percentile, in relatively few iterations, though care must be taken to ensure correct convergence.

Willmot and Lin state for x sufficiently large and m_1, m_2 integers satisfying $\alpha < m_1 < m_2$, upper bounds produced by a partition of m_1 α_j's are tighter than those of m_2. Though fairly weak, it suggests m should be small, to reduce computation time and generate tighter bounds for uppermost percentiles. Usually we define $m = -[-\alpha]$ to be the smallest integer greater than or equal to α. We derive values of κ and θ for specific loss distributions in their relevant sections.

Willmot and Lin also show their upper bound is sharp for integer α with exponential loss distributions (i.e. Pascal-exponential model). This demonstrates the bound is practical in at least one special case. Section 2 and 3 further investigate situations in which the upper bound is sufficiently tight to be of practical use.

Of principal interest however, is the 'partition' of α into $\alpha_1, \ldots, \alpha_m$ where each $\alpha_j \in (0,1]$ and sum to α. These α_j together with θ are used to calculate in turn, the coefficients C_α and weights $\overline{R}_i$ of the Erlang distributions that define the bound.

Willmot and Lin define two choices for the values of α_j when $m = -[-\alpha]$. The first is to choose all α_j's equal ($\alpha_j = \alpha / m$ for $j = 1,2,\ldots,m$). The second is to choose the first $m - 1$ α_j's equal to 1 ($\alpha_j = 1$ for $j = 1,2,\ldots,m - 1$ and $\alpha_m = \alpha - [\alpha]$). Willmot and Lin (1997b) also show the second choice produces the tightest upper bounds for all $x \geq 0$ when $\theta = \phi$, and for sufficiently large x when $\theta > \phi$.

We can generalise their choices by partitioning α into $m = -[-\alpha]$ partitions such that the first k α_j's are set to 1 and the remaining $m - k$ α_j's equal to a constant. When $k = m - 1$ we obtain the first case, and when $k = 0$, the second.

Extensive numerical testings suggests that these are the 'best' partitions in that for each partition, there exists a range of x for which $\overline{G}(x)$ is as tight as possible.

We now proceed to give numerical illustrations of Willmot and Lin's method.

Chapter 2

Loss Distributions

In examining different loss distributions and Willmot and Lin's upper bound, we restrict our examination to distributions having expected value of unity. All distributions can be scaled in this manner by changing the units of claim amounts.

We also wish to examine distributions of different variance and classes and for this reason have chosen to examine exponential, gamma and mixed-exponential loss distributions in detail.

A distribution function is said to be decreasing (decreasing) failure rate or DFR (IFR) if $\bar{F}(x+y)/\bar{F}(y)$ is nonincreasing (nondecreasing) in y for fixed $x \geq 0$.

By Willmot and Lin (2001) we note that exponential loss distributions are both IFR and DFR (§2.1), mixed-exponential loss distributions are DFR (§2.1.1) and gamma loss distributions are DFR (IFR) if $\alpha \leq (\geq)\ 1$ (§2.1.2)

2.1. Degenerate

We first investigate the degenerate case, $Y \sim$ Constant(1). Though degenerate loss distributions rarely occur in practice and exact methods of calculating them exist, it acts as an introduction to Willmot and Lin's method.

If $Y \sim$ Constant(1) then $\frac{1}{\phi} = e^{\kappa}$, so $\kappa = -\ln(\phi)$. Also, for any discrete distribution

$$\frac{1}{\theta} = \inf_{z \geq 0, \bar{F}(z) > 0} \frac{\sum_z^{\infty} e^{\kappa y} f(y)}{e^{\kappa z} \bar{F}(z)} = \frac{e^{\kappa y_{\max}} f(y_{\max})}{e^{\kappa y_{\max}} \bar{F}(y_{\max})} = 1,$$

so we obtain $\theta = 1$.

Also, if $\alpha = m$ is integer then $\alpha_j = 1$ for all j. In addition, $\theta = 1$, so $C_1 = 1$, and

$$\sum_{j=0}^{m} r_j z^j = \prod_{i=1}^{m} (1 - C_1 + C_1 z) = z^m.$$

Thus, $r_j = 0$ for $j = 0,1,\ldots,m-1$ and $r_m = 1$. Thus $\bar{R}_j = 1$ for all j and

$$\bar{G}(x) \leq e^{-\kappa x} \left(\sum_{i=1}^{m-1} \frac{(\kappa x)^i}{i!} \right) = \frac{\Gamma(m, \kappa x)}{\Gamma(m)}, \quad x \geq 0,$$

or that $\bar{G}(x)$ is bounded by the tail distribution of an Erlang(m, κ) distribution.

If α is fractional, our analysis is more complicated and is easier shown in the form of a numerical example. Let $\alpha = 10.5$, $\phi = 0.5$ and

$$\{a_j\} = \{1, 1, 1, 1, 1, 1, 0.9, 0.9, 0.9, 0.9, 0.9\}.$$

Then from above, $\kappa = -\ln(\phi) = 0.69315$ and $\theta = 1$. Futhermore

$$C_1 = \tfrac{1}{0.5}\left(1 - (1-0.5)^1\right) = 1 \text{ and } C_{0.9} = \tfrac{1}{0.5}\left(1 - (1-0.5)^{0.9}\right) = 0.928227,$$

so

$$\begin{aligned}\sum_{j=0}^{m} r_j z^j &= \prod_{i=1}^{6} (1 - C_1 + C_1 z) \prod_{i=1}^{5} (1 - C_{0.9} + C_{0.9} z) \\ &= z^6 (0.071773 + 0.928227z)^5 \\ &= 0.000002z^6 + 0.000123z^7 + 0.003186z^8 + 0.041199z^9 \\ &\quad +0.266408z^{10} + 0.689082z^{11},\end{aligned}$$

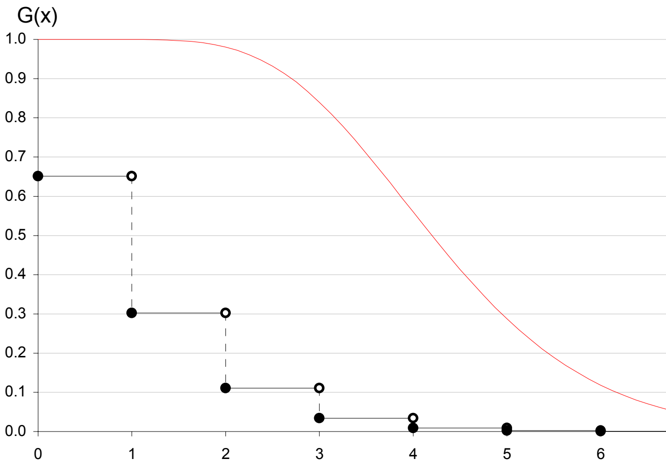
and finally

$$\{\bar{R}_j\} = \{1, 1, 1, 1, 1, 0.999998, 0.999875, 0.996689, 0.955490, 0.689082\}.$$

The following pages contain numerical comparisons between the "exact" aggregate distribution, as calculated by Panjer's methiod, and the upper bound, as calculated bu Willmot and Lin's method. Included with each graph are selected tabulated figures of the aggregate distribution as well as uppermost percentiles obtained under each method.

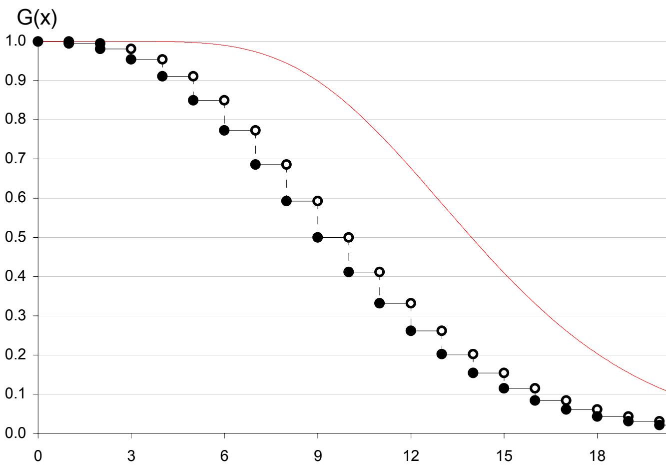
The comparisons are divided into pages exhibiting the trends and effects of ϕ (figures 2.1.1 – 2.1.3), α (figures 2.1.4 – 2.1.6) and fractional α (figures 2.1.7 – 2.1.9). Since $\theta > \phi$, when α is fractional, we have a family of optimal upper bounds each corresponding to a different value of k, the number of 1's in our partition. The numerical illustrations show only bounds and figures corresponding to $k = 10$, 6 and 0 and tabulated figures shown in grey are non-optimal bounds.

Effect of ϕ on upper bounds



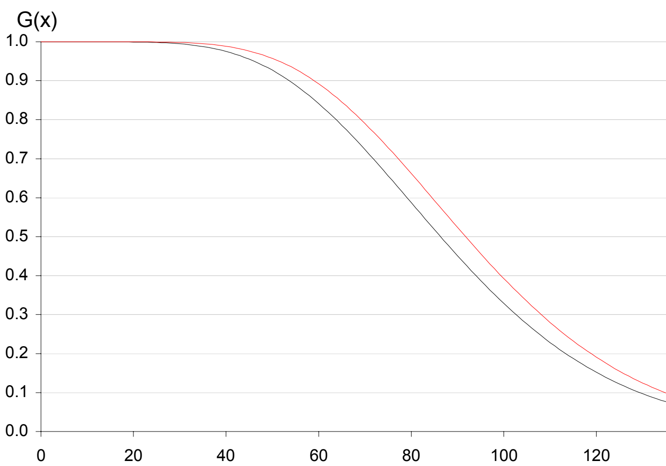
x	Panjer	W&L
0	0.65132	1.00000
2	0.11087	0.98034
4	0.00923	0.55972
6	0.00050	0.11843
8	0.00002	0.01223
10	0.00000	0.00079
Percentiles		
0.900	3.000	6.170
0.950	3.000	6.821
0.990	4.000	8.157
0.995	5.000	8.685

Figure 2.1.1: $\phi = 0.1$, $\alpha = 10$



x	Panjer	W&L
0	0.99023	1.00000
5	0.84912	0.99690
10	0.41190	0.83738
15	0.11476	0.40932
20	0.02139	0.11605
25	0.00299	0.02201
Percentiles		
0.900	16.000	20.494
0.950	18.000	22.658
0.990	23.000	27.098
0.995	24.000	28.852

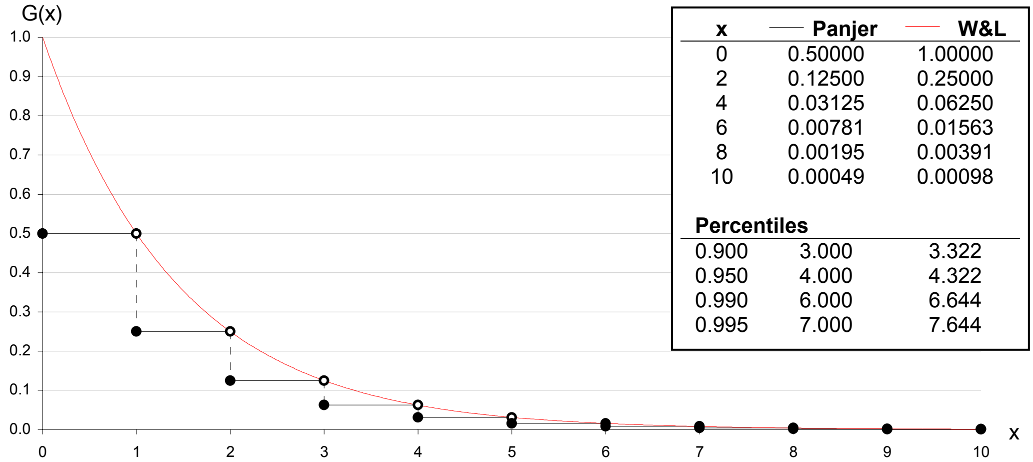
Figure 2.1.2: $\phi = 0.5$, $\alpha = 10$



x	Panjer	W&L
0	1.00000	1.00000
50	0.92693	0.95738
100	0.32899	0.39290
150	0.03590	0.04766
200	0.00187	0.00265
Percentiles		
0.900	130.000	134.832
0.950	144.000	149.062
0.990	173.000	178.275
0.995	185.000	189.809

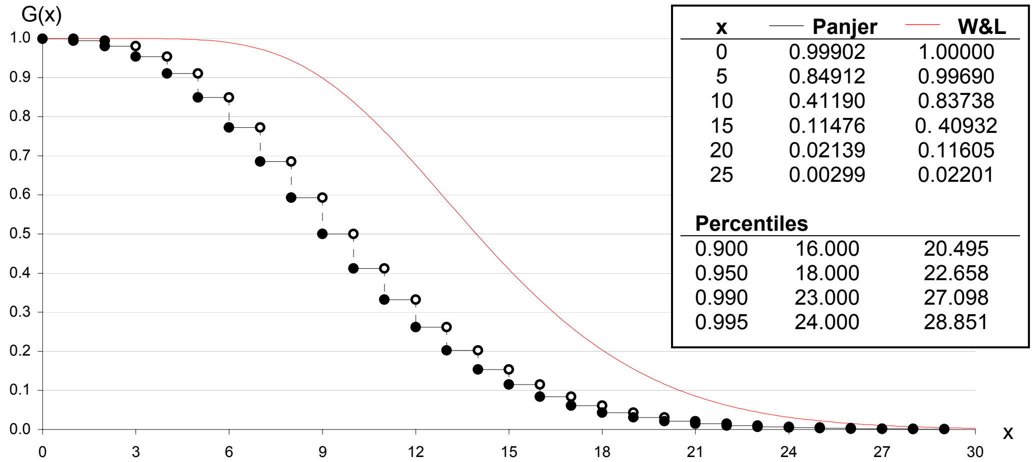
Figure 2.1.3: $\phi = 0.9$, $\alpha = 10$

Effect of α on upper bounds



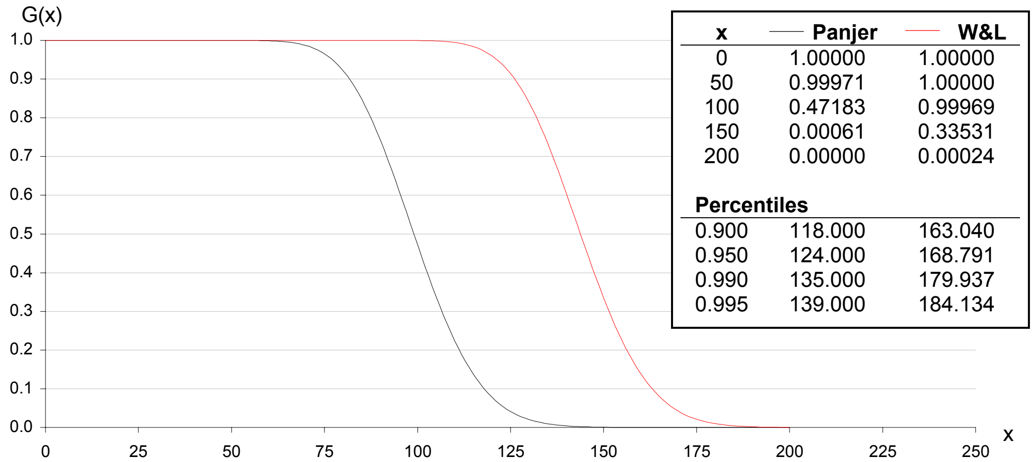
x	Panjer	W&L
0	0.50000	1.00000
2	0.12500	0.25000
4	0.03125	0.06250
6	0.00781	0.01563
8	0.00195	0.00391
10	0.00049	0.00098
Percentiles		
0.900	3.000	3.322
0.950	4.000	4.322
0.990	6.000	6.644
0.995	7.000	7.644

Figure 2.1.4: $\phi = 0.5$, $\alpha = 1$



x	Panjer	W&L
0	0.99902	1.00000
5	0.84912	0.99690
10	0.41190	0.83738
15	0.11476	0. 40932
20	0.02139	0.11605
25	0.00299	0.02201
Percentiles		
0.900	16.000	20.495
0.950	18.000	22.658
0.990	23.000	27.098
0.995	24.000	28.851

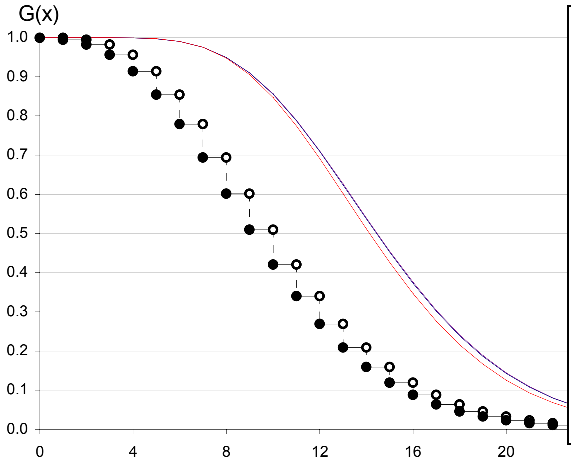
Figure 2.1.5: $\phi = 0.5$, $\alpha = 10$



x	Panjer	W&L
0	1.00000	1.00000
50	0.99971	1.00000
100	0.47183	0.99969
150	0.00061	0.33531
200	0.00000	0.00024
Percentiles		
0.900	118.000	163.040
0.950	124.000	168.791
0.990	135.000	179.937
0.995	139.000	184.134

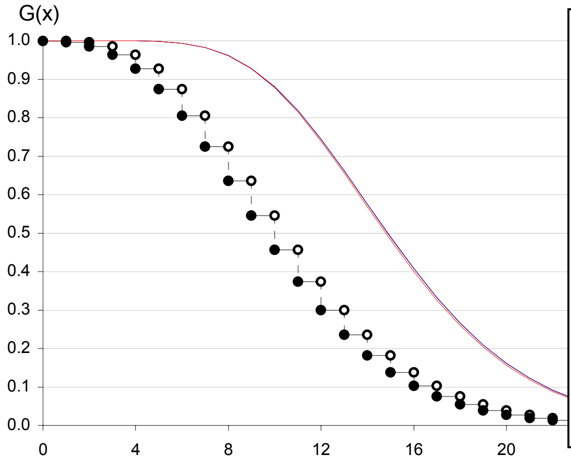
Figure 2.1.6: $\phi = 0.5$, $\alpha = 100$

Effect of fractional α on upper bounds



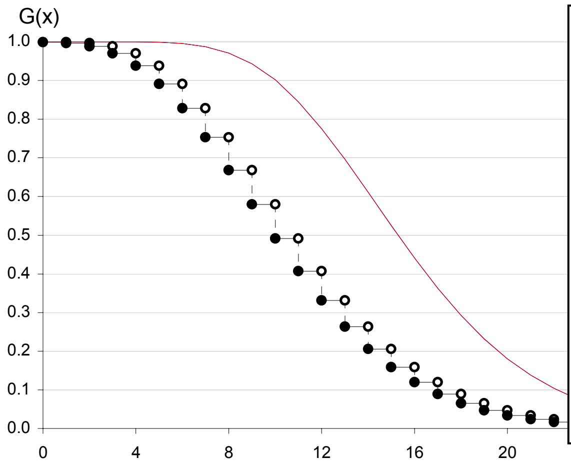
x	Panj	k=10	k=6	k=0
0	0.999	1.000	1.000	1.000
1	0.994	1.000	1.000	1.000
…	…	…	…	…
6	0.780	0.990	0.990	0.990
7	0.694	0.975	0.975	0.975
8	0.602	0.948	0.949	0.950
9	0.509	0.905	0.910	0.910
…	…	…	…	…
29	0.001	0.005	0.007	0.007
30	0.000	0.004	0.005	0.005
Percentiles				
0.900	16.000	20.763	21.246	21.291
0.950	18.000	22.952	23.486	23.534
0.990	23.000	27.443	28.071	28.123
0.995	24.000	29.216	29.876	29.930

Figure 2.1.7: $\phi = 0.5$, $\alpha = 10.1$



x	Panj	k=10	k=6	k=0
0	0.999	1.000	1.000	1.000
1	0.996	1.000	1.000	1.000
…	…	…	…	…
6	0.805	0.993	0.993	0.993
7	0.724	0.982	0.982	0.982
8	0.636	0.961	0.962	0.962
9	0.545	0.927	0.928	0.928
…	…	…	…	…
29	0.001	0.008	0.008	0.008
30	0.001	0.005	0.006	0.006
Percentiles				
0.900	17.000	21.260	21.703	21.716
0.950	19.000	23.501	23.946	23.960
0.990	23.000	28.087	28.535	28.551
0.995	25.000	29.893	30.343	30.358

Figure 2.1.8: $\phi = 0.5$, $\alpha = 10.5$



x	Panj	k=10	k=6	k=0
0	0.999	1.000	1.000	1.000
1	0.997	1.000	1.000	1.000
…	…	…	…	…
6	0.828	0.996	0.996	0.996
7	0.753	0.988	0.988	0.988
8	0.668	0.971	0.971	0.971
9	0.580	0.943	0.943	0.943
…	…	…	…	…
29	0.001	0.010	0.010	0.010
30	0.001	0.007	0.007	0.007
Percentiles				
0.900	17.000	22.122	22.126	22.127
0.950	19.000	24.366	24.370	24.371
0.990	24.000	28.957	28.962	28.962
0.995	26.000	30.765	30.769	30.770

Figure 2.1.9: $\phi = 0.5$, $\alpha = 10.9$

From figures 2.1.1 – 2.1.3 we see a definite trend that as $\phi\uparrow$, Willmot and Lin's method generates tighter bounds. With $\alpha = 10$ and $\phi = 0.1$, their method overestimates the 99.5th percentile by 73.7% compared to only 2.6% when $\phi = 0.9$.

From figures 2.1.4 – 2.1.6 our upper bound is slightly tighter when α is small. In fact when $\alpha = 1$, Willmot and Lin's method gives the tightest continuous bound possible for our discrete loss distribution (when $\alpha = 1$ our model is compound geometric and Willmot and Lin's upper bound is exponential). In contrast, when $\alpha = 100$ Willmot and Lin's method overestimates the 99.5th percentile by 32.7%.

From figures 2.1.7 – 2.1.9 we see Willmot and Lin's (1997b) statement that " [$k = m$] or [$k = 0$] appear to be best" is incorrect. Though the former is best for large x and the latter for small x, there is a small range where a transition occurs. For practical purposes however, the former partition seems to suffice for all $x \geq 0$.

When the fractional part of α is small, the multiple upper bounds obtained differ significantly, converging as the fractional part increases until α again takes integer value. Our upper bound also seems slightly weaker when α is near integer. When $\phi = 0.5$ and $\alpha = 10.5$, Willmot and Lin's method overestimates the 99.5th percentile by 21.6% compared to 22.1% for $\alpha = 10.1$ and $\alpha = 10.9$.

In summary, Willmot and Lin's method seems practical only for large values of ϕ, with a small improvement if α is small. In particular when $\alpha = 1$, their method can be modified to give an exact bound. Fractional values of α make a negligible improvement to Willmot and Lin's bounds and for practical purposes, the partition with $\alpha_j = 1$ for $j = 1,2,\ldots,m - 1$ seems sufficient for all $x \geq 0$.

2.2. Exponential

Exponential loss distributions often arise naturally in insurance applications, and are very readily adopted in research papers due to their simplicity. Panjer & Willmot (1981) show the compound negative binomial-exponential model is equivalent to a compound binomial-exponential model, and as a result aggregate claims and net stop-loss premiums can be calculated exactly as finite sums.

Willmot and Lin (1997b) show their bound to be sharp for integer α and exponential loss distribution, and by direct comparison to Panjer & Willmot (1981), we see their results are identical.

If Y ~ Exponential(β) then $f(y) = \beta e^{-\beta y}$ for y ≥ 0, so

$$, \overline{F}(z) = \int_z^\infty \beta e^{-\beta y}\,dy = e^{-\beta z},$$

and for $z \geq 0$

$$\int_z^\infty e^{\kappa y}\,dF(y) = \int_z^\infty e^{\kappa y}\beta e^{-\beta y}\,dy = \frac{\beta e^{-(\beta-\kappa)z}}{\beta-\kappa}.$$

When $z = 0$ we obtain $\frac{1}{\phi} = \left(\frac{\beta}{\beta-\kappa}\right)$ from which $\kappa = \beta(1-\phi)$. Then for $z \geq 0$

$$\frac{1}{\theta} = \inf_{z\geq 0, \overline{F}(z)>0} \frac{\beta e^{-(\beta-\kappa)y}}{(\beta-\kappa)e^{\kappa z}e^{-\beta z}} = \frac{\beta}{\beta-\kappa} = \frac{1}{\phi},$$

so we obtain $\theta = \phi$

As in the degenerate case if $\alpha = m$ is integer then $\alpha_j = 1$ for all j and as $C_1 = \theta$

$$\sum_{j=0}^{m} r_j z^j = \prod_{i=1}^{m}(1 - C_1 + C_1 z) = (1-\theta+\theta z)^{\alpha}.$$

Thus, $r_j = \binom{\alpha}{j}\theta^j(1-\theta)^{\alpha-j}$ for $j = 0,1,\ldots,\alpha$ where $\theta = \phi$. This leads to

$$\overline{G}(x) \leq e^{-\beta(1-\phi)x}\sum_{i=1}^{\alpha-1}\frac{(\beta(1-\phi)x)^i}{i!}\sum_{j=i+1}^{\alpha}\binom{\alpha}{j}\phi^j(1-\phi)^{\alpha-j}, \quad x \geq 0.$$

By evaluating the compound Pascal-exponential distribution (Willmot and Lin 2001), §4.1), we see this inequality is sharp. Interchanging the order of summation

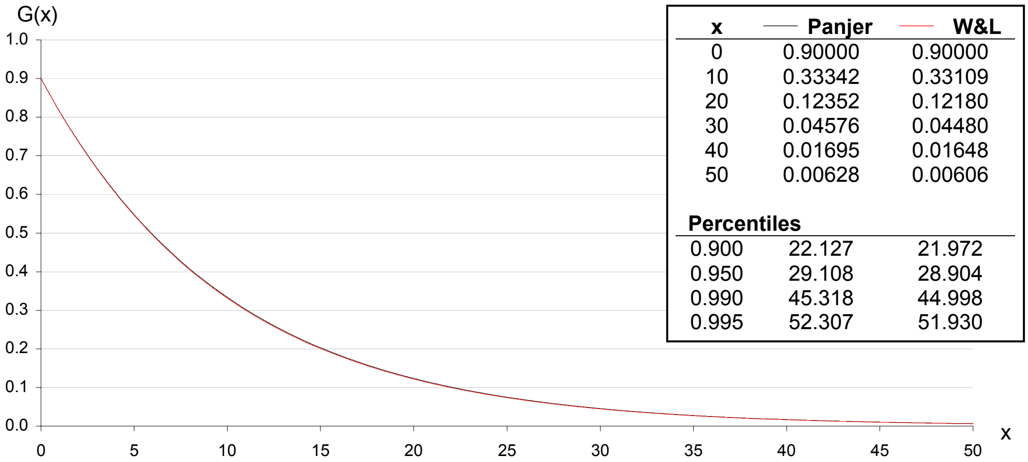
$$\bar{G}(x)=\sum_{j=1}^{\alpha}\binom{\alpha}{j}\phi^{j}(1-\phi)^{\alpha-j}\sum_{i=0}^{j-1}\frac{\left(\beta(1-\phi)x\right)^{i}e^{-\beta(1-\phi)x}}{i!},\quad x\geq 0,$$

and we obtain Panjer & Willmot's (1981) result, which states that $G(x)$ follows a compound binomial-exponential model with counting distribution Binomial(α, κ) and loss distribution Exponential(β). This is a generalisation of Lundberg's (1933) Geometric-exponential model, since if $\alpha = 1$

$$\bar{G}(x)=\phi e^{-\beta(1-\phi)x},\quad x\geq 0.$$

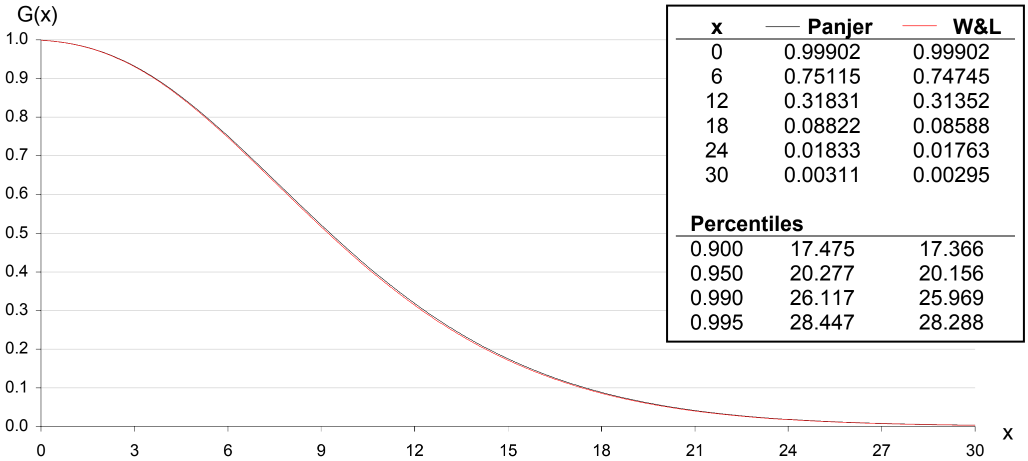
Included next are numerical comparisons between the "exact" aggregate tail distribution and Willmot and Lin's upper bound, exhibiting the trends and effects of ϕ and α (figures 2.2.1 – 2.2.3) and fractional α (figures 2.2.4 – 2.2.6) respectively. For exponential loss distributions we have $\theta = \phi$, so there is a unique optimal upper bound when α takes fractional values.

Effect of ϕ and α on upper bounds



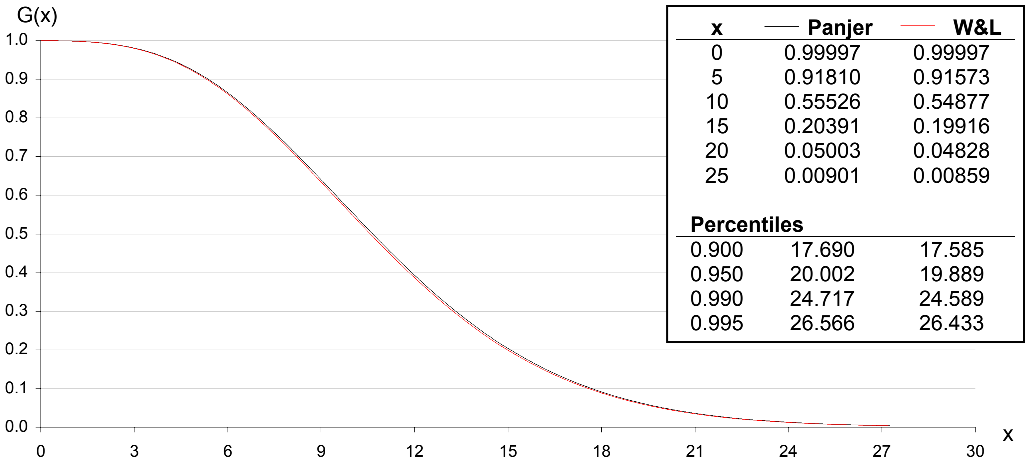
x	Panjer	W&L
0	0.90000	0.90000
10	0.33342	0.33109
20	0.12352	0.12180
30	0.04576	0.04480
40	0.01695	0.01648
50	0.00628	0.00606
Percentiles		
0.900	22.127	21.972
0.950	29.108	28.904
0.990	45.318	44.998
0.995	52.307	51.930

Figure 2.2.1: $\phi = 0.9$, $\alpha = 1$



x	Panjer	W&L
0	0.99902	0.99902
6	0.75115	0.74745
12	0.31831	0.31352
18	0.08822	0.08588
24	0.01833	0.01763
30	0.00311	0.00295
Percentiles		
0.900	17.475	17.366
0.950	20.277	20.156
0.990	26.117	25.969
0.995	28.447	28.288

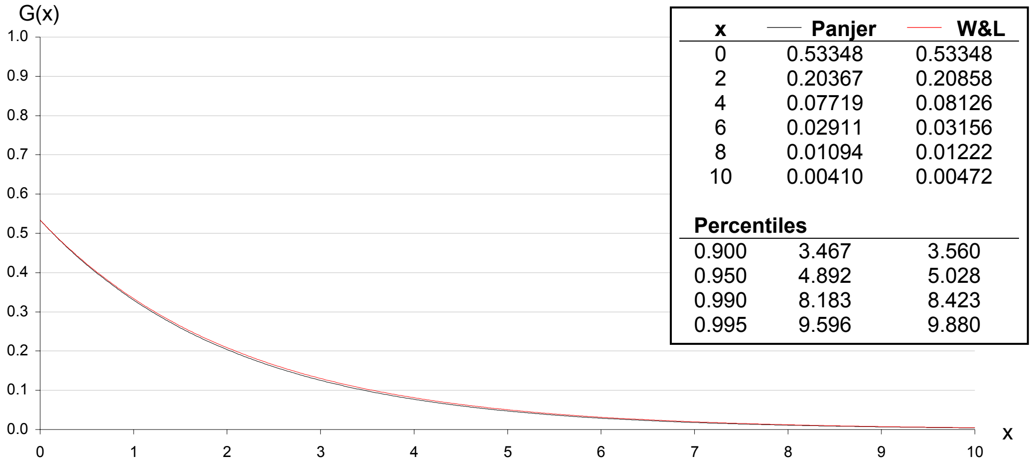
Figure 2.2.2: $\phi = 0.5$, $\alpha = 10$



x	Panjer	W&L
0	0.99997	0.99997
5	0.91810	0.91573
10	0.55526	0.54877
15	0.20391	0.19916
20	0.05003	0.04828
25	0.00901	0.00859
Percentiles		
0.900	17.690	17.585
0.950	20.002	19.889
0.990	24.717	24.589
0.995	26.566	26.433

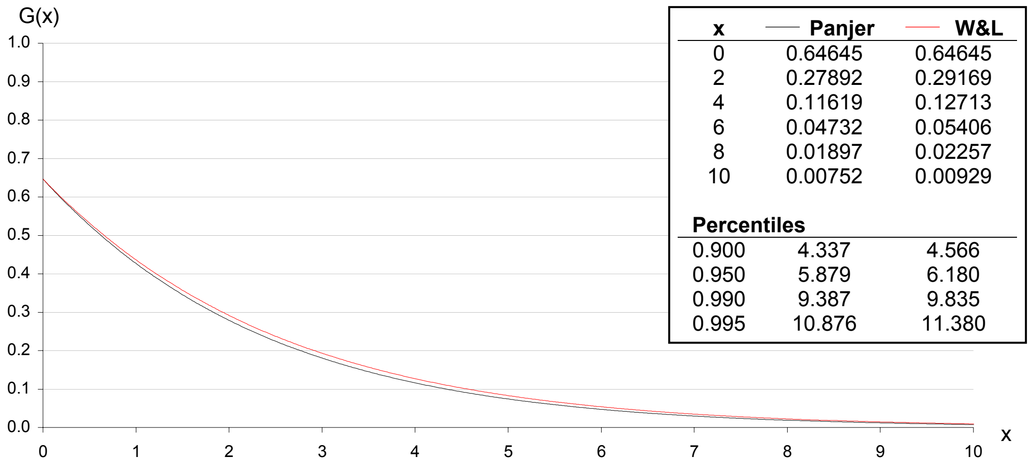
Figure 2.2.3: $\phi = 0.1$, $\alpha = 100$

Effect of fractional α on upper bounds



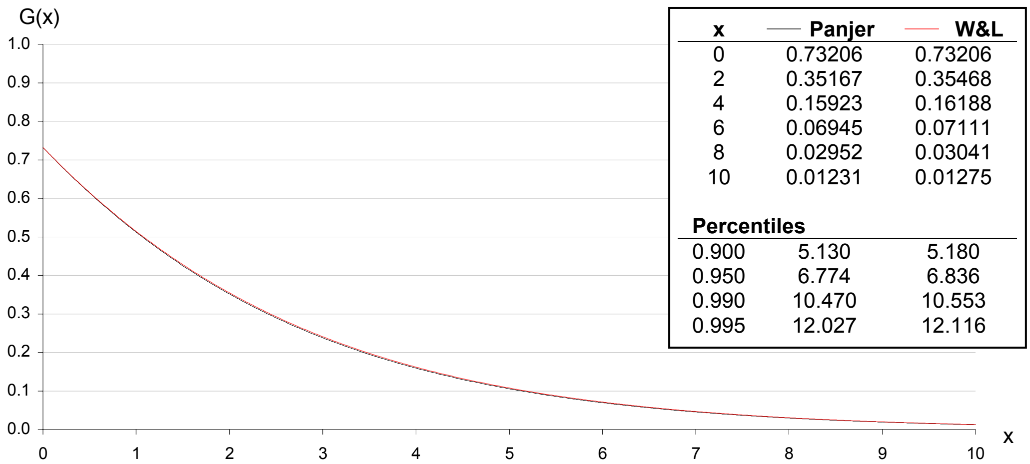
x	Panjer	W&L
0	0.53348	0.53348
2	0.20367	0.20858
4	0.07719	0.08126
6	0.02911	0.03156
8	0.01094	0.01222
10	0.00410	0.00472
Percentiles		
0.900	3.467	3.560
0.950	4.892	5.028
0.990	8.183	8.423
0.995	9.596	9.880

Figure 2.2.4: $\phi = 0.5$, $\alpha = 1.1$



x	Panjer	W&L
0	0.64645	0.64645
2	0.27892	0.29169
4	0.11619	0.12713
6	0.04732	0.05406
8	0.01897	0.02257
10	0.00752	0.00929
Percentiles		
0.900	4.337	4.566
0.950	5.879	6.180
0.990	9.387	9.835
0.995	10.876	11.380

Figure 2.2.5: $\phi = 0.5$, $\alpha = 1.5$



x	Panjer	W&L
0	0.73206	0.73206
2	0.35167	0.35468
4	0.15923	0.16188
6	0.06945	0.07111
8	0.02952	0.03041
10	0.01231	0.01275
Percentiles		
0.900	5.130	5.180
0.950	6.774	6.836
0.990	10.470	10.553
0.995	12.027	12.116

Figure 2.2.6: $\phi = 0.5$, $\alpha = 1.9$

Figures 2.2.1 – 2.2.3 demonstrate Willmot and Lin's (1997b) result that their bound is exact for compound Pascal-exponential models. In our examples the bounds lie below those calculated by Panjer's recursion due to the discretisation of the loss distribution (see section 1.1).

Figures 2.2.4 – 2.2.6 use the result that as $\theta = \phi$, the best partition for all x occurs when $m = -[-\alpha]$, with $\alpha_j = 1$ for $j = 1,2,\ldots,m-1$ and $\alpha_m = \alpha - [\alpha]$ (Willmot and Lin, 1997b). The results suggest Willmot and Lin's bounds are weakened slightly by fractional α, however still seem tight enough for practical purpose. Their methods perform better for near integer values of α, with large fractional parts performing better than small fractional parts.

When $\phi = 0.5$ and $\alpha = 1.1$, their bound overestimates the 99.5^{th} percentile by 3.0%. This overestimation rises to 4.6% when $\alpha = 1.5$, though falls back down to 0.7% by the time $\alpha = 1.9$. This pattern seems to hold for all α, though becomes less significant as α gets larger. When $\alpha = 10.1$ and 10.9, Willmot and Lin's generates 99.5^{th} percentiles more accurately than those generated by Panjer's methods and only overestimates by 0.2% when $\alpha = 10.5$.

In summary, for exponential loss distributions, Willmot and Lin's method is exact for integer values of α and all values of ϕ, and its use is recommended for such situations, due to its sharpness and (especially for large α and large ϕ) speed of computationWillmot and Lin's bounds are slightly weakened by fractional α, but for practical purposes, this difference is negligible.

2.3. Gamma

Erlang loss distributions can arise in insurance applications if each claim amount is the sum of exponentially distributed claims. For storm insurance, the aggregate claim size of a portfolio of *r* insured properties in a local region, where each property is susceptible to the same storms and claim amounts follow an Exponential(β) distribution, would follow an Erlang(*r*, β) distribution.

In a similar manner, Gamma loss distributions may arise in insurance applications. Though they have no natural interpretation like their Erlang counterparts, they are more flexible and may be used ot fit empirical loss data.

If $Y \sim$ Gamma(r, β), $f(y) = \frac{\beta^r}{\Gamma(r)} y^{r-1} e^{-\beta y}$ for $y \geq 0$, then

$$\bar{F}(z) = \frac{\beta}{\Gamma(r)} \int_z^\infty (\beta y)^r e^{-\beta y} dy = \frac{\Gamma(r, \beta z)}{\Gamma(r)},$$

where Γ(r, βz) is the incomplete (upper) gamma function and

$$\int_z^\infty e^{\kappa y} dF(y) = \frac{\beta}{\Gamma(r)} \int_z^\infty (\beta y)^r e^{-(\beta-\kappa)y} dy = \left(\frac{\beta}{\beta-\kappa}\right)^r \frac{\Gamma(r, (\beta-\kappa)z)}{\Gamma(r)}.$$

When $z = 0$, we obtain $\frac{1}{\phi} = \left(\frac{\beta}{\beta-\kappa}\right)^r$ from which $\kappa = \beta\left(1-\phi^{\frac{1}{r}}\right)$. Then for $z \geq 0$

$$\frac{1}{\theta} = \left(\frac{\beta}{\beta-\kappa}\right)^r \inf_{z\geq 0, \bar{F}(z)>0} \frac{\Gamma(r, (\beta-\kappa)z)}{e^{\kappa z}\Gamma(r, \beta z)}.$$

Now for $r > 1$

$$e^{-\kappa z}\Gamma(r, (\beta-\kappa)z) = \int_{(\beta-\kappa)z}^\infty t^{r-1} e^{-(t+\kappa z)} dt = \int_{\beta z}^\infty (t-\kappa z)^{r-1} e^{-t} dt \leq \Gamma(r, \beta z),$$

with equality occurring only at $z = 0$. As the inequality is more severe for larger z the infimum above occurs as $z \to \infty$. Also notice from the inequality that

$$e^{-\kappa z}\Gamma(r, (\beta-\kappa)z) = \left(\frac{c-\kappa z}{c}\right)^{r-1} \Gamma(r, \beta z),$$

for some c, $\beta z \le c < \infty$. Thus as $z \to \infty$, $c \to \beta z$ and so

$$\lim_{z\to\infty} e^{-\kappa z}\Gamma\left(r,(\beta-\kappa)z\right) = \left(\frac{\beta-\kappa}{\beta}\right)^{r-1}\Gamma\left(r,\beta z\right).$$

Thus

$$\frac{1}{\theta} = \left(\frac{\beta}{\beta-\kappa}\right)^{r} \lim_{z\to\infty}\frac{\Gamma\left(r,(\beta-\kappa)z\right)}{e^{\kappa z}\Gamma\left(r,\beta z\right)} = \left(\frac{\beta}{\beta-\kappa}\right) = \left(\frac{1}{\phi}\right)^{\frac{1}{r}}, \quad r \ge 1.$$

Similarly when $r < 1$ we obtain $e^{-\kappa z}\Gamma\left(r,(\beta-\kappa)z\right) \ge \Gamma\left(r,\beta z\right)$ with equality occurring only at $z = 0$, thus infimum above occurs at $z = 0$, so

$$\frac{1}{\theta} = \left(\frac{\beta}{\beta-\kappa}\right)^{r} \lim_{z\to 0}\frac{\Gamma\left(r,(\beta-\kappa)z\right)}{e^{\kappa z}\Gamma\left(r,\beta z\right)} = \left(\frac{\beta}{\beta-\kappa}\right)^{r} = \frac{1}{\phi}, \quad r \le 1.$$

Thus $\theta = \begin{cases} \phi^{\frac{1}{r}}, & r \ge 1 \\ \phi, & r \le 1 \end{cases}$, with $r = 1$ corresponding to the exponential case.

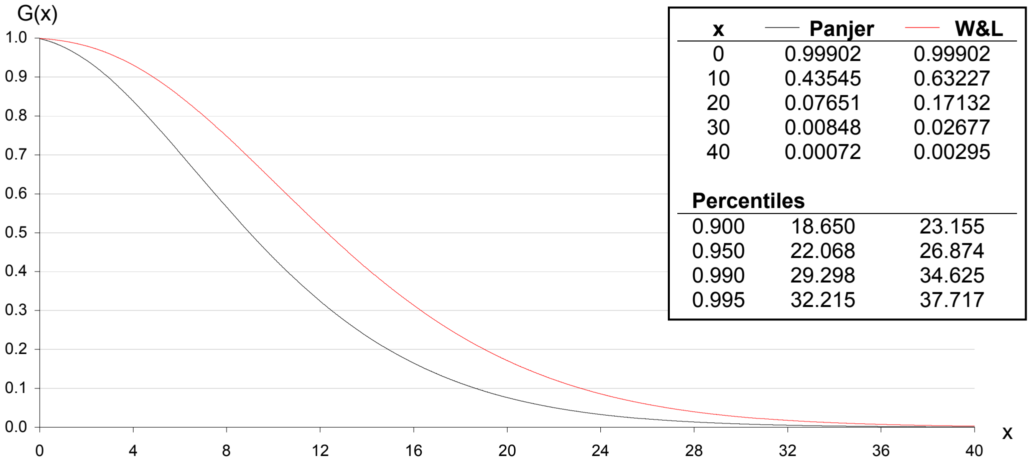
If $\alpha = m$ is an integer, again we have $\alpha_j = 1$ and $C_1 = \theta$. So our values of r_j are identical to the exponential case with $r_j = \binom{\alpha}{j}\theta^j(1-\theta)^{\alpha-j}$, thus

$$\bar{G}(x) \le e^{-\beta\left(1-\phi^{\frac{1}{r}}\right)x}\sum_{i=1}^{\alpha-1}\frac{\left(\beta\left(1-\phi^{\frac{1}{r}}\right)x\right)^{i}}{i!}\sum_{j=i+1}^{\alpha}\binom{\alpha}{j}\theta^j(1-\theta)^{\alpha-j}, \quad x \ge 0,$$

where $\theta = \begin{cases} \phi^{\frac{1}{r}}, & r \ge 1 \\ \phi, & r \le 1 \end{cases}$.

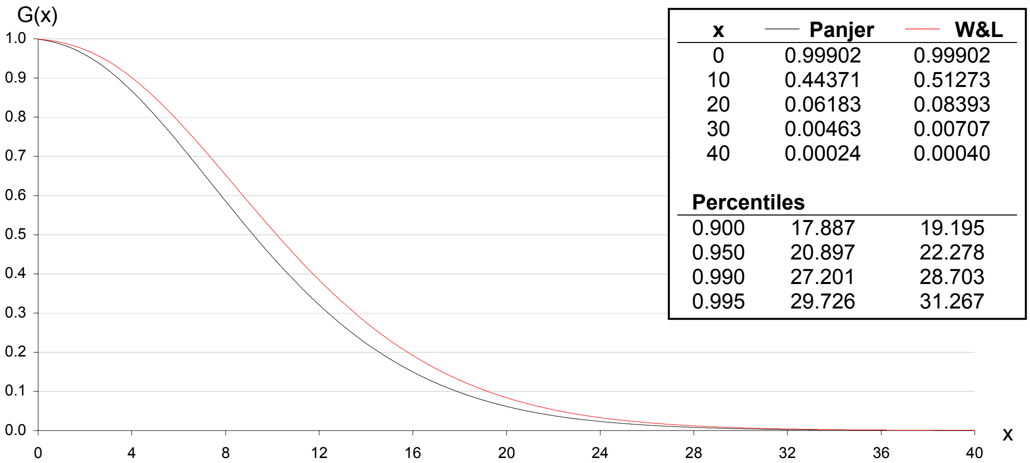
Numerical comparisons between the "exact" tail distribution and Willmot and Lin's upper bound exhibit similar trends for values of ϕ and α a for the exponential case, and for this reason are omitted. Figures 2.3.1 – 2.3.12 show the effect of r and fractional α, when $r < 1$ and $r > 1$ respectively.

Effect of *r* on upper bounds (when *r* < 1)



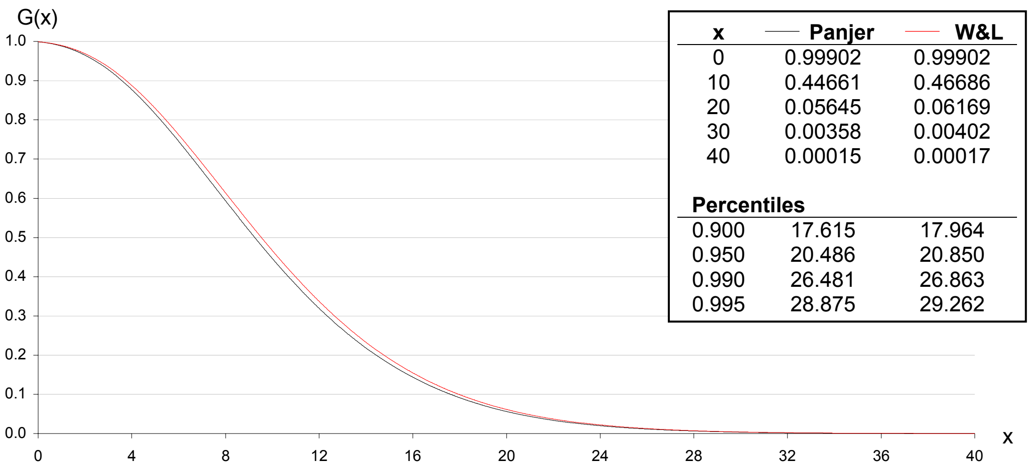
x	Panjer	W&L
0	0.99902	0.99902
10	0.43545	0.63227
20	0.07651	0.17132
30	0.00848	0.02677
40	0.00072	0.00295
Percentiles		
0.900	18.650	23.155
0.950	22.068	26.874
0.990	29.298	34.625
0.995	32.215	37.717

Figure 2.3.1: $r = 0.5$, $\phi = 0.5$, $\alpha = 10$



x	Panjer	W&L
0	0.99902	0.99902
10	0.44371	0.51273
20	0.06183	0.08393
30	0.00463	0.00707
40	0.00024	0.00040
Percentiles		
0.900	17.887	19.195
0.950	20.897	22.278
0.990	27.201	28.703
0.995	29.726	31.267

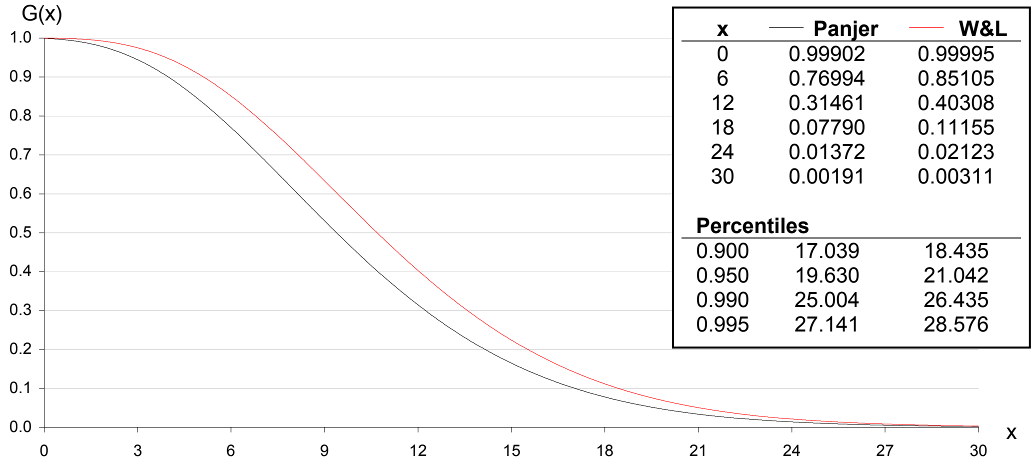
Figure 2.3.2: $r = 0.75$, $\phi = 0.5$, $\alpha = 10$



x	Panjer	W&L
0	0.99902	0.99902
10	0.44661	0.46686
20	0.05645	0.06169
30	0.00358	0.00402
40	0.00015	0.00017
Percentiles		
0.900	17.615	17.964
0.950	20.486	20.850
0.990	26.481	26.863
0.995	28.875	29.262

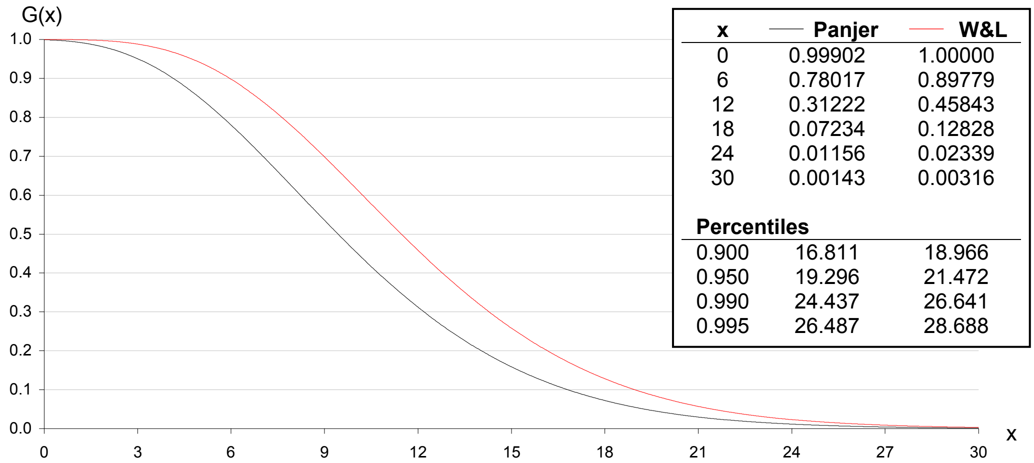
Figure 2.3.3: $r = 0.9$, $\phi = 0.5$, $\alpha = 10$

Effect of *r* on upper bounds (when *r* > 1)



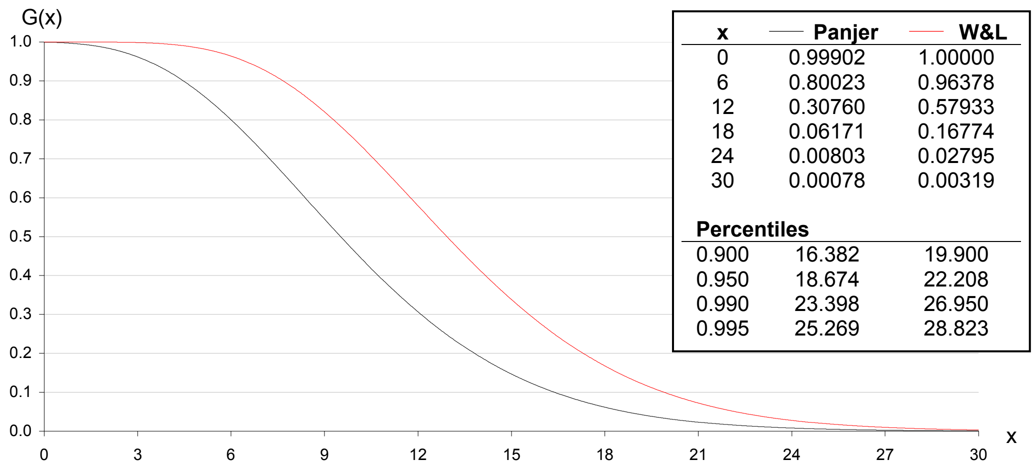
x	Panjer	W&L
0	0.99902	0.99995
6	0.76994	0.85105
12	0.31461	0.40308
18	0.07790	0.11155
24	0.01372	0.02123
30	0.00191	0.00311
Percentiles		
0.900	17.039	18.435
0.950	19.630	21.042
0.990	25.004	26.435
0.995	27.141	28.576

Figure 2.3.4: $r = 1.5$, $\phi = 0.5$, $\alpha = 10$



x	Panjer	W&L
0	0.99902	1.00000
6	0.78017	0.89779
12	0.31222	0.45843
18	0.07234	0.12828
24	0.01156	0.02339
30	0.00143	0.00316
Percentiles		
0.900	16.811	18.966
0.950	19.296	21.472
0.990	24.437	26.641
0.995	26.487	28.688

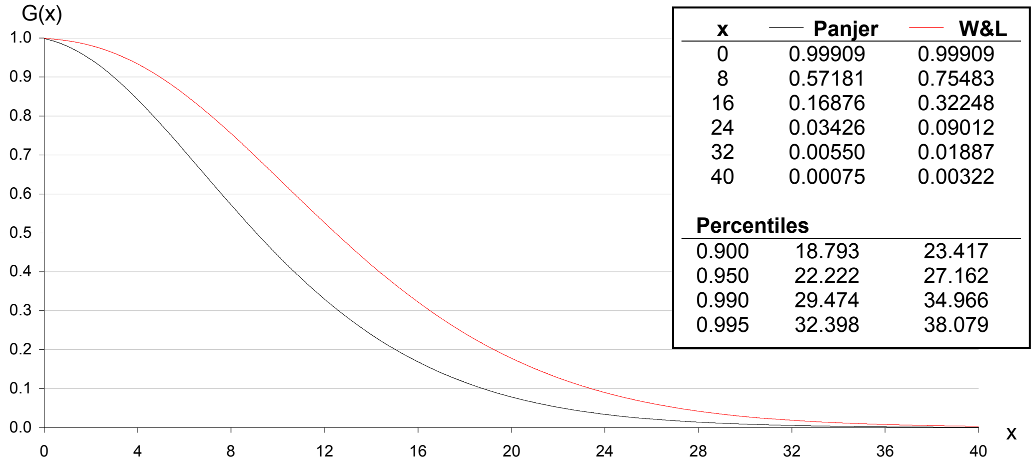
Figure 2.3.5: $r = 2$, $\phi = 0.5$, $\alpha = 10$



x	Panjer	W&L
0	0.99902	1.00000
6	0.80023	0.96378
12	0.30760	0.57933
18	0.06171	0.16774
24	0.00803	0.02795
30	0.00078	0.00319
Percentiles		
0.900	16.382	19.900
0.950	18.674	22.208
0.990	23.398	26.950
0.995	25.269	28.823

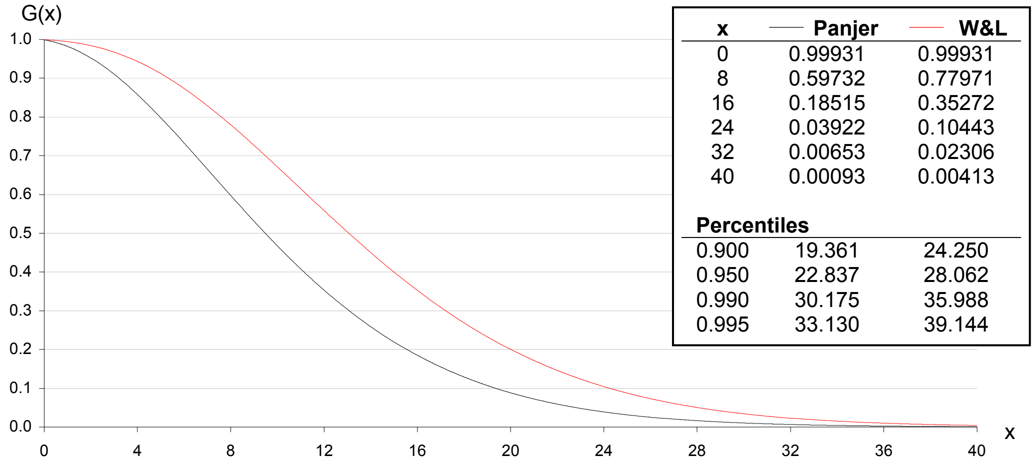
Figure 2.3.6: $r = 5$, $\phi = 0.5$, $\alpha = 10$

Effect of fractional α on upper bounds (when $r < 1$)



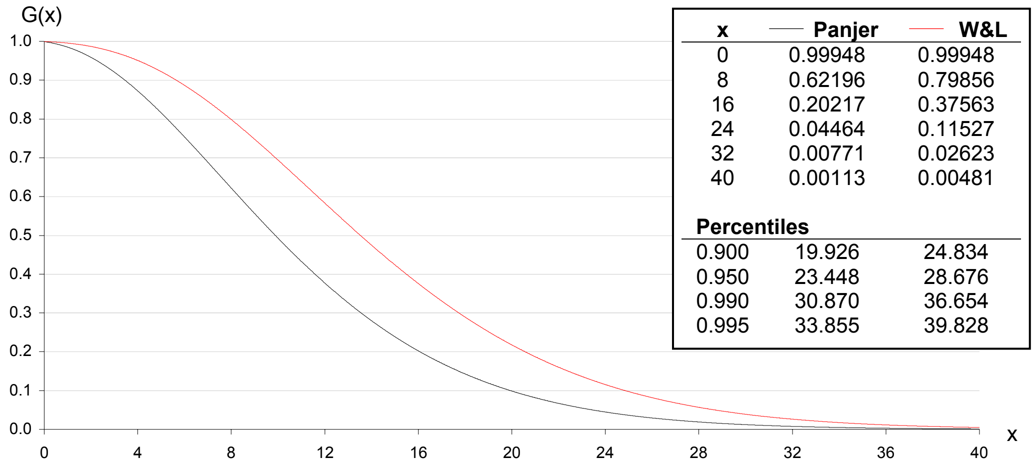
x	Panjer	W&L
0	0.99909	0.99909
8	0.57181	0.75483
16	0.16876	0.32248
24	0.03426	0.09012
32	0.00550	0.01887
40	0.00075	0.00322
Percentiles		
0.900	18.793	23.417
0.950	22.222	27.162
0.990	29.474	34.966
0.995	32.398	38.079

Figure 2.3.7: $r = 0.5$, $\phi = 0.5$, $\alpha = 10.1$



x	Panjer	W&L
0	0.99931	0.99931
8	0.59732	0.77971
16	0.18515	0.35272
24	0.03922	0.10443
32	0.00653	0.02306
40	0.00093	0.00413
Percentiles		
0.900	19.361	24.250
0.950	22.837	28.062
0.990	30.175	35.988
0.995	33.130	39.144

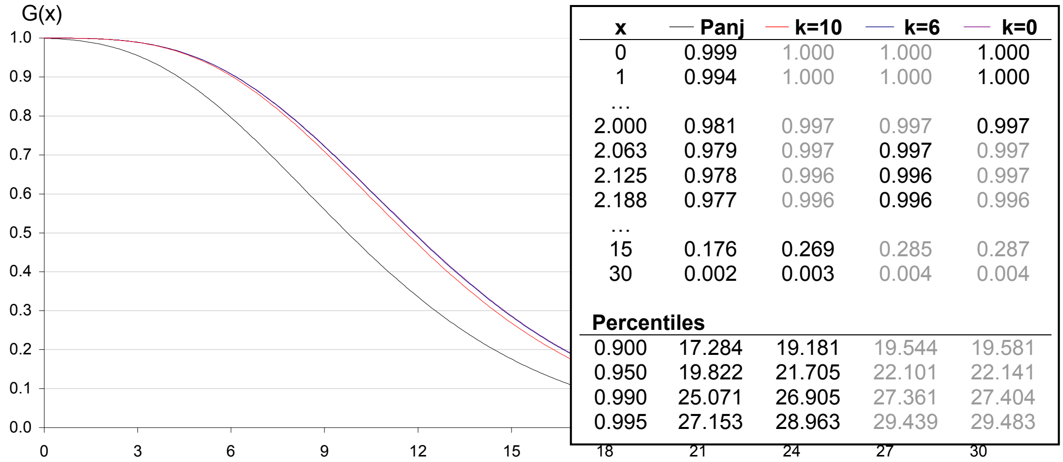
Figure 2.3.8: $r = 0.5$, $\phi = 0.5$, $\alpha = 10.5$



x	Panjer	W&L
0	0.99948	0.99948
8	0.62196	0.79856
16	0.20217	0.37563
24	0.04464	0.11527
32	0.00771	0.02623
40	0.00113	0.00481
Percentiles		
0.900	19.926	24.834
0.950	23.448	28.676
0.990	30.870	36.654
0.995	33.855	39.828

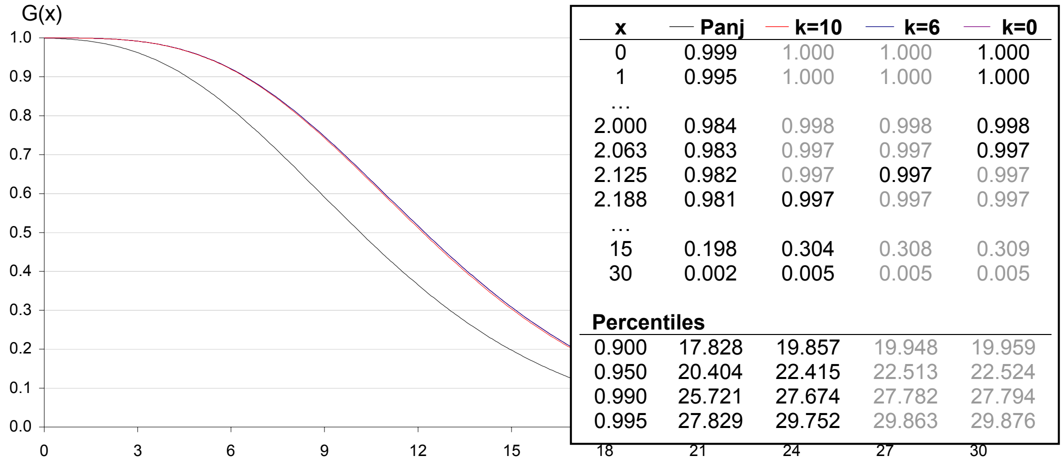
Figure 2.3.9: $r = 0.5$, $\phi = 0.5$, $\alpha = 10.9$

Effect of fractional α on upper bounds (when *r* > 1)



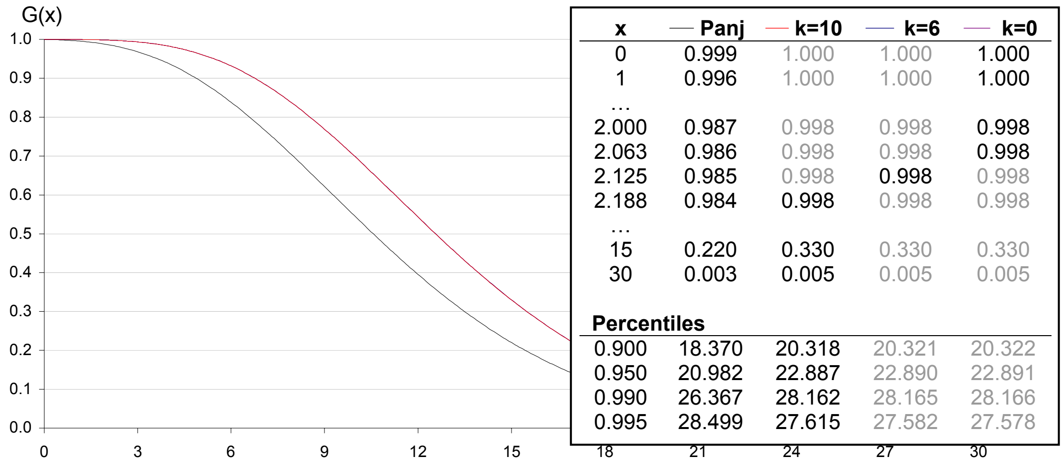
x	Panj	k=10	k=6	k=0
0	0.999	1.000	1.000	1.000
1	0.994	1.000	1.000	1.000
…				
2.000	0.981	0.997	0.997	0.997
2.063	0.979	0.997	0.997	0.997
2.125	0.978	0.996	0.996	0.997
2.188	0.977	0.996	0.996	0.996
…				
15	0.176	0.269	0.285	0.287
30	0.002	0.003	0.004	0.004
Percentiles				
0.900	17.284	19.181	19.544	19.581
0.950	19.822	21.705	22.101	22.141
0.990	25.071	26.905	27.361	27.404
0.995	27.153	28.963	29.439	29.483

Figure 2.3.10: $r = 2$, $\phi = 0.5$, $\alpha = 10.1$



x	Panj	k=10	k=6	k=0
0	0.999	1.000	1.000	1.000
1	0.995	1.000	1.000	1.000
…				
2.000	0.984	0.998	0.998	0.998
2.063	0.983	0.997	0.997	0.997
2.125	0.982	0.997	0.997	0.997
2.188	0.981	0.997	0.997	0.997
…				
15	0.198	0.304	0.308	0.309
30	0.002	0.005	0.005	0.005
Percentiles				
0.900	17.828	19.857	19.948	19.959
0.950	20.404	22.415	22.513	22.524
0.990	25.721	27.674	27.782	27.794
0.995	27.829	29.752	29.863	29.876

Figure 2.3.11: $r = 2$, $\phi = 0.5$, $\alpha = 10.5$



x	Panj	k=10	k=6	k=0
0	0.999	1.000	1.000	1.000
1	0.996	1.000	1.000	1.000
…				
2.000	0.987	0.998	0.998	0.998
2.063	0.986	0.998	0.998	0.998
2.125	0.985	0.998	0.998	0.998
2.188	0.984	0.998	0.998	0.998
…				
15	0.220	0.330	0.330	0.330
30	0.003	0.005	0.005	0.005
Percentiles				
0.900	18.370	20.318	20.321	20.322
0.950	20.982	22.887	22.890	22.891
0.990	26.367	28.162	28.165	28.166
0.995	28.499	27.615	27.582	27.578

Figure 2.3.12: $r = 2$, $\phi = 0.5$, $\alpha = 10.9$

2.4. Mixture of exponentials

An even more flexible family of distributions is the mixture of exponentials, or hyperexponential, distributions. By Berstein's Theorem (see Feller, 1971) any monotonic distribution function can be approximated arbitrarily closely by a mixture of exponentials and many papers such as Asmussen, Nerman and Olsson (1996) and Feldmann and Whitt (1998) are devoted to such algorithms. Mixture of exponential distributions are therefore exptensively employed in insurance applications to fit a loss distribution to empirical claim data.

If Y is a mixture of exponentials given by $f(y) = \sum_{i=1}^{n} w_i \beta_i e^{-\beta_i y}$ for $y \geq 0$, then

$$\overline{F}(z) = \int_z^\infty \sum_{i=1}^{n} w_i \beta_i e^{-\beta_i y} dy = \sum_{i=1}^{n} w_i e^{-\beta_i z},$$

and

$$\int_z^\infty e^{\kappa y} dF(y) = \int_z^\infty e^{\kappa y} \sum_{i=1}^{n} w_i \beta_i e^{-\beta_i y} dy = e^{\kappa z} \sum_{i=1}^{n} w_i \left(\frac{\beta_i}{\beta_i - \kappa} \right) e^{-\beta_i z}.$$

When $z = 0$, we obtain $\frac{1}{\phi} = \sum_{i=1}^{n} w_i \left(\frac{\beta_i}{\beta_i - \kappa} \right)$ which is easiest solved numerically.

The above summation converges to 1 as $\kappa \to 0$ and to ∞ as $\kappa \to \beta_{\min}$ so our desired solution occurs within the range $0 < \kappa < \beta_{\min}$. Also note the summation is dominated by the term with minimum denominator (i.e. $\beta_{\min} - \kappa > 0$). Thus $\frac{1}{\phi} \approx \left(\frac{\beta_{\min}}{\beta_{\min} - \kappa} \right)$ so $\kappa \approx \beta_{\min}(1 - \phi)$.

Thus for $z \geq 0$

$$\frac{1}{\theta} = \inf_{z \geq 0, \overline{F}(z) > 0} \frac{\sum_{i=1}^{n} w_i \left(\frac{\beta_i}{\beta_i - \kappa} \right) e^{-\beta_i z}}{\sum_{i=1}^{n} w_i e^{-\beta_i z}}.$$

Now, the numerator decreases at a a faster rate than the denominator since

$$-\frac{d}{dz}\sum_{i=1}^{n} w_i\left(\frac{\beta_i}{\beta_i-\kappa}\right)e^{-\beta_i z} \ge \left(\frac{\beta_{\max}}{\beta_{\max}-\kappa}\right)\sum_{i=1}^{n} w_i\beta_i e^{-\beta_i z} > \sum_{i=1}^{n} w_i\beta_i e^{-\beta_i z},$$

so the function is monotonically increasing and infinum occurs at $z = 0$

$$\frac{1}{\theta} = \lim_{z\to 0}\frac{\sum_{i=1}^{n} w_i\left(\frac{\beta_i}{\beta_i-\kappa}\right)e^{-\beta_i z}}{\sum_{i=1}^{n} w_i e^{-\beta_i z}} = \frac{\sum_{i=1}^{n} w_i\left(\frac{\beta_i}{\beta_i-\kappa}\right)}{\sum_{i=1}^{n} w_i} = \frac{1}{\phi},$$

so we obtain $\theta = \phi$.

If $\alpha = m$ integer, then since $\theta = \phi$ our values of r_j are again identical to the exponential case. So $r_j = \binom{\alpha}{j}\theta^j(1-\theta)^{\alpha-j}$ and thus

$$\bar{G}(x) \le e^{-\kappa x}\sum_{i=1}^{\alpha-1}\frac{(\kappa x)^i}{i!}\sum_{j=i+1}^{\alpha}\binom{\alpha}{j}\phi^j(1-\phi)^{\alpha-j}, \quad x \ge 0,$$

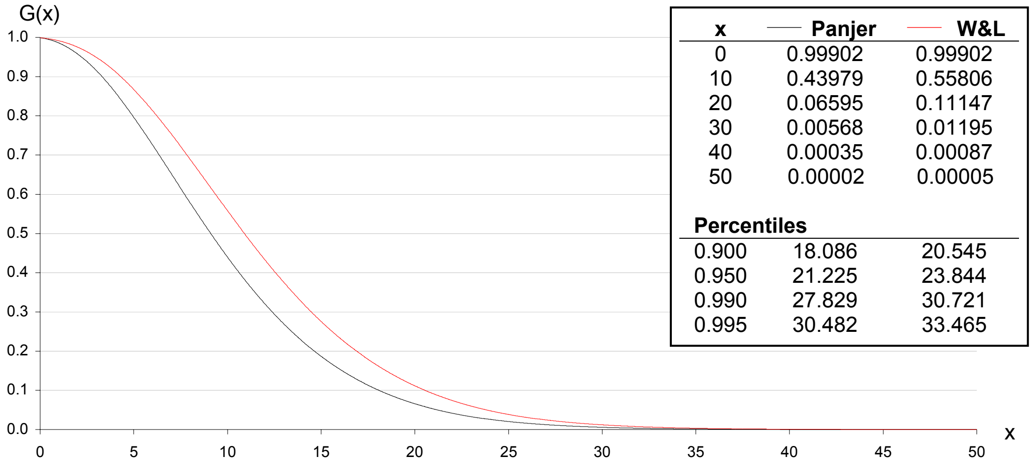
where κ is the numerical solution of

$$\frac{1}{\phi} = \sum_{i=1}^{n} w_i\left(\frac{\beta_i}{\beta_i-\kappa}\right).$$

If we consider mixtures of two exponentials then κ can be solved explicitly as the solution of a quadratic. Considering only two exponentials also enables us to systematically investigate the parameters as there is only 2 free variables, $w_1 \ge 0$ and $\beta_1 \ge 1$.

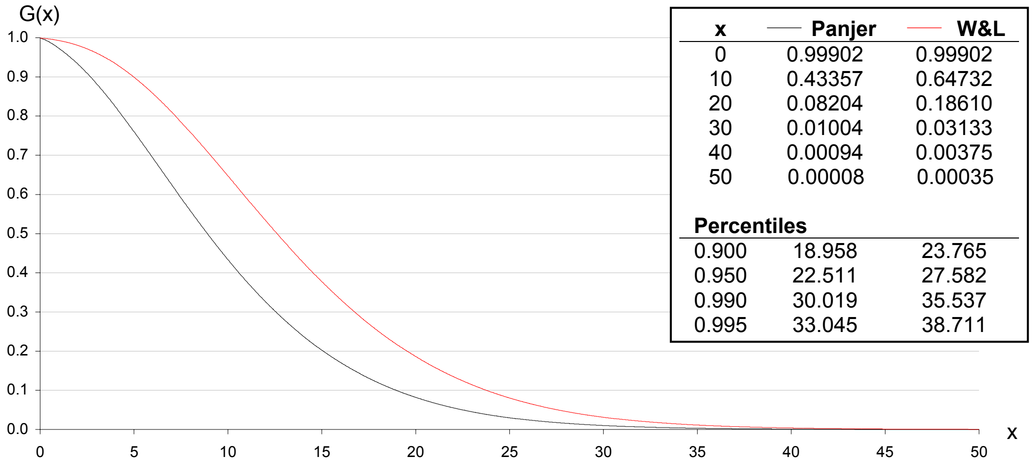
Figures 2.4.1 – 2.4.3 and 2.4.4. – 2.4.6 show the effects of w_1 and β_1 respectively on Willmot and Lin's upper bounds. The effect of ϕ and α are similar to previous cases and are omitted. Also included in each figure is a calculation of the variance of the mixture of exponentials.

Effect of β_1 on upper bounds



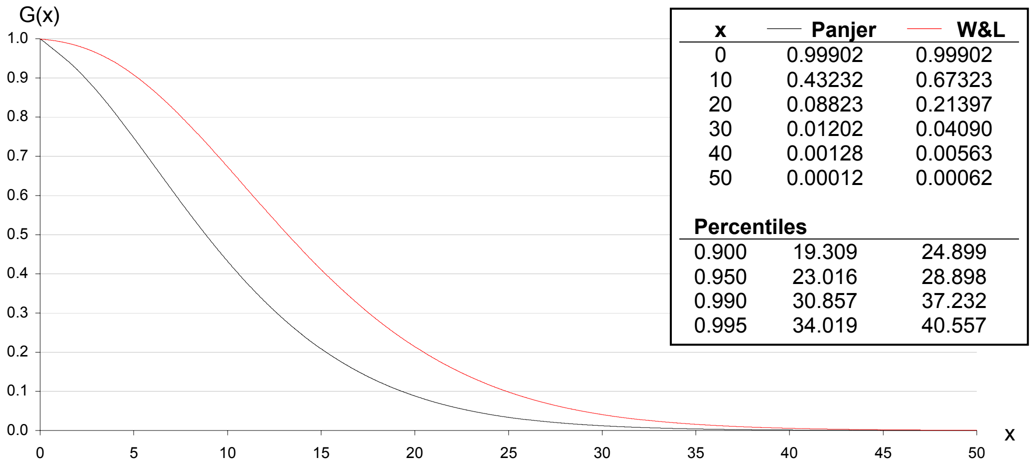
x	Panjer	W&L
0	0.99902	0.99902
10	0.43979	0.55806
20	0.06595	0.11147
30	0.00568	0.01195
40	0.00035	0.00087
50	0.00002	0.00005
Percentiles		
0.900	18.086	20.545
0.950	21.225	23.844
0.990	27.829	30.721
0.995	30.482	33.465

Figure 2.4.1: $w_1 = 0.5$, $\beta_1 = 2$, $\phi = 0.5$, $\alpha = 10$ (Variance = 1.5)



x	Panjer	W&L
0	0.99902	0.99902
10	0.43357	0.64732
20	0.08204	0.18610
30	0.01004	0.03133
40	0.00094	0.00375
50	0.00008	0.00035
Percentiles		
0.900	18.958	23.765
0.950	22.511	27.582
0.990	30.019	35.537
0.995	33.045	38.711

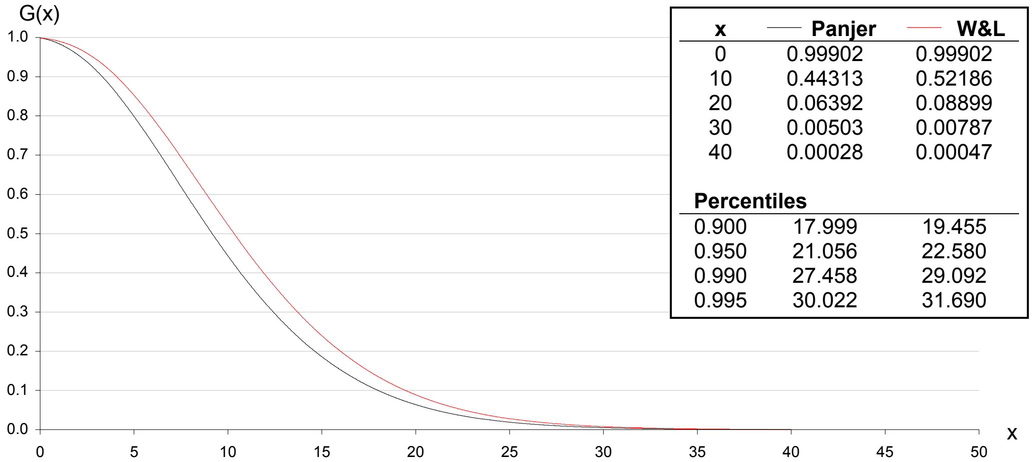
Figure 2.4.2: $w_1 = 0.5$, $\beta_1 = 5$, $\phi = 0.5$, $\alpha = 10$ (Variance = 2.28)



x	Panjer	W&L
0	0.99902	0.99902
10	0.43232	0.67323
20	0.08823	0.21397
30	0.01202	0.04090
40	0.00128	0.00563
50	0.00012	0.00062
Percentiles		
0.900	19.309	24.899
0.950	23.016	28.898
0.990	30.857	37.232
0.995	34.019	40.557

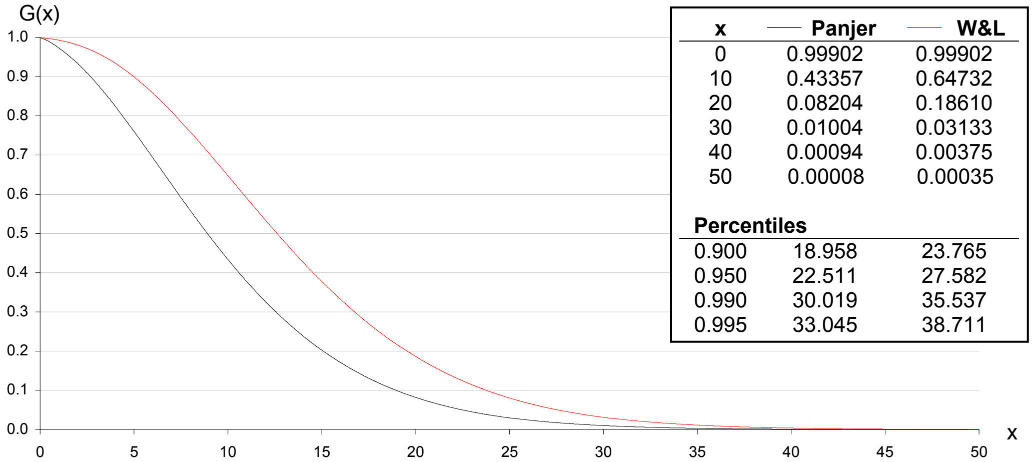
Figure 2.4.3: $w_1 = 0.5$, $\beta_1 = 10$, $\phi = 0.5$, $\alpha = 10$ (Variance = 15.58)

Effect of w_1 on upper bounds



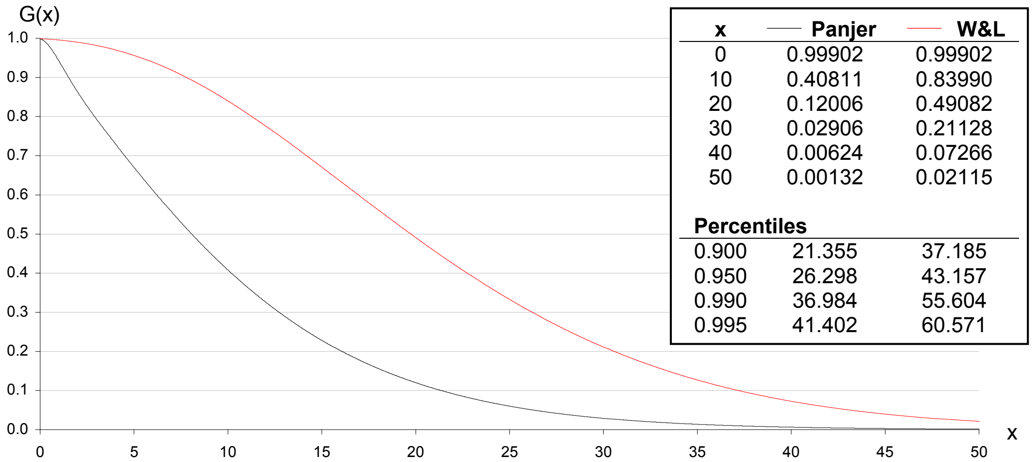
x	Panjer	W&L
0	0.99902	0.99902
10	0.44313	0.52186
20	0.06392	0.08899
30	0.00503	0.00787
40	0.00028	0.00047
Percentiles		
0.900	17.999	19.455
0.950	21.056	22.580
0.990	27.458	29.092
0.995	30.022	31.690

Figure 2.4.4: $w_1 = 0.25$, $\beta_1 = 5$, $\phi = 0.5$, $\alpha = 10$ (Variance = 1.42667)



x	Panjer	W&L
0	0.99902	0.99902
10	0.43357	0.64732
20	0.08204	0.18610
30	0.01004	0.03133
40	0.00094	0.00375
50	0.00008	0.00035
Percentiles		
0.900	18.958	23.765
0.950	22.511	27.582
0.990	30.019	35.537
0.995	33.045	38.711

Figure 2.4.5: $w_1 = 0.5$, $\beta_1 = 5$, $\phi = 0.5$, $\alpha = 10$ (Variance = 2.28)



x	Panjer	W&L
0	0.99902	0.99902
10	0.40811	0.83990
20	0.12006	0.49082
30	0.02906	0.21128
40	0.00624	0.07266
50	0.00132	0.02115
Percentiles		
0.900	21.355	37.185
0.950	26.298	43.157
0.990	36.984	55.604
0.995	41.402	60.571

Figure 2.4.6: $w_1 = 0.75$, $\beta_1 = 5$, $\phi = 0.5$, $\alpha = 10$ (Variance = 4.84)

Similarly to previous cases, Willmot and Lin's bounds are tighter for large ϕ and smaller, fractional values of α. With equal weights, $\beta_1 = 5$, $\alpha = 10$, their method overestimates the 99.5th percentile by 25.0% compared to just 2.5% when $\phi = 0.1$ and 0.9 respectively, 26.9% compared to just 8.0% when $\alpha = 100$ and 1 respectively and 17.1% compared to 15.2% when $\alpha = 10$ and 10.5 respectively.

Figures 2.4.1 – 2.4.3 show that as the mixture of exponentials distribution approaches exponential (i.e. $\beta_1 \to 1$), Willmot and Lin's bounds approach exact bounds. For equal weights, $\phi = 0.5$ and $\alpha = 10$, the overestimation of the 99.5th percentile dropped from 19.2% when $\beta_1 = 10$ to 9.8% when $\beta_1 = 2$.

Figures 2.4.4 – 2.4.6 show a similar trend as $w_1 \to 0$. However as $w_1 \to 1$, the bounds are weaken enormously. For $\beta_1 = 5$, $\phi = 0.5$ and $\alpha = 10$, the 99.5th percentile overestimation was 17.1% when $w_1 = 0.5$ compared to just 5.6% when $w_1 = 0.25$ and 46.3% when $w_1 = 0.75$.

These trends are explained since in extreme cases, the numerical solution approaches that of exponential with parameter β_{min}. As $\beta_1 \to 1$ or $w_1 \to 0$, β_{min} tends to 1 and so the bound approaches the exact exponential case, whereas as $w_1 \to 1$, β_{min} tends to 0 and the bound approaches a horizontal line.

A final trend obtained from figures 2.4.1 – 2.4.6 is that Willmot and Lin's bound appears tighter when the variance of the mixture of exponentials is small, although this does not hold absolutely.

In summary, for mixed-exponential loss distributions, Willmot and Lin's bounds are tighter for large values of ϕ, with small improvements for smaller α. However, the tightest bounds are obtained when the mixture of exponentials (more specifically, variance) is close to that of exponential(β_{min}).

Chapter 3

Insurance Applications

3.1 Reserve Calculations

We are often interested in calculating the value of required reservices, u, the amount of additional funds required, which in addition to premium income exceeds the aggregate claim amount for a given probability, say 95%. That is

$$\Pr\left(S < u + P\right) \geq 0.95,$$

where $P = (1 + \theta)E[X]$ for loading level, say, $\theta = 0.2$ and $E[X] = \alpha\phi(1 - \phi)$.

We may use Willmott and Lin's method to generate an upper bound for the required aggregate percentile and hence an upper bound for the required reserve, while the "true" reserve may be obtained via Panjer's method. The excess of the upper bound is the *cost* of Willmot and Lin's method.

Table 1 below illustrate some typical examples. The first two examples have large values of α, and so from our Section 2 results, Willmot and Lin's bounds are quire weak (99.5^{th} percentile is overestimated by 25.1% and 14.3% respectively). In contrast, the last two examples have large values of ϕ and so have quite tight bounds (1.9% and 0.5% respectively).

	Panjer	W&L	'Cost'	
			Absolute	Relative
Gamma(0.5, 0.5) (ϕ = 0.5, α = 100) – 25.1% overestimation				
90.0%	7.157	43.531	36.374	508.2%
95.0%	15.595	53.011	37.416	239.9%
99.0%	32.237	71.544	39.307	121.9%
99.5%	38.616	78.575	39.959	103.5%
Gamma(2.0, 2.0) (ϕ = 0.5, α = 100) – 14.3% overestimation				
90.0%	1.539	21.984	20.445	1328%
95.0%	7.993	28.555	20.562	257.3%
99.0%	20.582	41.324	20.742	100.8%
99.5%	25.362	46.156	20.783	81.95%
Gamma(0.5, 0.5) (ϕ = 0.9, α = 10) – 1.9% overestimation				
90.0%	14.795	18.814	40.19	27.16%
95.0%	30.599	34.562	3.963	12.95%
99.0%	63.134	66.907	3.773	5.98%
99.5%	76.045	79.684	3.639	4.78%
Gamma(2.0, 2.0) (ϕ = 0.9, α = 10) – 0.5% overestimation				
90.0%	11.906	13.367	1.461	12.27%
95.0%	26.616	27.967	1.351	5.08%
99.0%	56.865	57.944	1.079	1.90%
99.5%	68.860	69.781	0.921	1.34%

Table 1: Required reserves

In all examples, the absolute costs seem fairly stable across the various probabilities, however there is a noticeable trend that the relative cost is significantly exaggerated for lower probabilities. This is simply caused by small values of u, in particular note the two largest relative cost figures, 502.8% and 1,328%, correspond with the two smallest values of u, 7.157 and 1.529 respectively.

However, even at 99.5% probability, the relative costs of reserving using Willmot and Lin's method seem a significantly greater thatn the overestimations of the 99.5th percentile from Section 2. This implies that in general, Willmot and Lin's method is required to be very tight (say, < 1% error in the 99.5th percentile) for the required reserves under their method to be of practial use.

3.2 Stop-loss reinsurance

By Panjer and Willmot (1981), the net stop-loss premium of stop loss level x is

$$R(x)=\int_x^\infty (y-x)\,dG(y)=\int_x^\infty \left(1-G(y)\right)dy.$$

Substituting Willmot and Lin's bound

$$\begin{aligned}R(x)&\le\int_x^\infty e^{-\kappa y}\sum_{i=0}^{m-1}\bar{R}_i\frac{(\kappa y)^i}{i!}dy\\&=\frac{1}{\kappa}\sum_{i=0}^{m-1}\bar{R}_i\int_x^\infty e^{-\kappa y}\frac{\kappa^{i+1}y^i}{i!}dy\\&=\frac{e^{-\kappa x}}{\kappa}\sum_{i=0}^{m-1}\bar{R}_i\sum_{j=0}^{i}\frac{(\kappa x)^j}{j!},\end{aligned}$$

since $\int_x^\infty e^{-\kappa y}\frac{\kappa^{i+1}y^i}{i!}dy$ is the incomplete (upper) Gamma function, $\frac{\Gamma(i,\kappa x)}{\Gamma(i)}$.

If evaluated efficiently, $R(x)$ consists of $2(m-1)$ terms. Rearranging further

$$\begin{aligned}R(x)&\le\frac{e^{-\kappa x}}{\kappa}\sum_{i=0}^{m-1}\sum_{k=i+1}^{m}r_k\sum_{j=0}^{i}\frac{(\kappa x)^j}{j!}\\&=\frac{e^{-\kappa x}}{\kappa}\sum_{k=1}^{m}r_k\sum_{j=0}^{k-1}\sum_{i=j}^{k-1}\frac{(\kappa x)^j}{j!}\\&=\frac{e^{-\kappa x}}{\kappa}\sum_{k=1}^{m}r_k\sum_{j=0}^{k-1}(k-j)\frac{(\kappa x)^j}{j!}.\end{aligned}$$

For the negative binomial-exponential model, $\bar{G}(x)$, and therefore $R(x)$ is exact and we have $r_j=\binom{\alpha}{j}\theta^j(1-\theta)^{\alpha-j}$ and $\kappa=\beta(1-\phi)$, so

$$R(x)=\frac{e^{-\beta(1-\phi)x}}{(1-\phi)\beta}\left[\sum_{k=1}^{\alpha}\binom{\alpha}{k}\phi^k(1-\phi)^{\alpha-k}\sum_{j=0}^{k-1}(k-j)\frac{\left(\beta(1-\phi)x\right)^j}{j!}\right],$$

which consists of $m(m-1)/2$ terms and by direct comparison is equivalent to Panjer and Willmot (1981). Thus we have an improved results, with an expression that can evaluate negative binomial-exponential model, stop-loss premiums in only $2(m-1)$ terms.

Note also when $\alpha = 1$ we obtain the geometric-exponential model result

$$R(x) = \frac{\phi e^{-\beta(1-\phi)x}}{(1-\phi)\beta}.$$

We can now apply our derived formula fo $R(x)$ to evaluate upper bounds for stop-loss premiums. This provides us with a fast, finite sum evaluation as opposed to Panjer's methods which sometimes produces an intractable infinite sum expression. Also, as the approximation overestimates the true net stop-loss premium, it can be used in practical situations with confidence.

Table 2 below illustrates three examples for net stop-loss premiums. The 99.5th percentile was overestimated by 26.77%, 1.33% and 5.28% respectively

	Panjer	W&L	%Error
Gamma(0.5, 0.5) – 26.88% overestimation			
R(5)	5.4322	8.5217	56.87%
R(10)	2.3145	4.6669	101.64%
R(15)	0.8112	2.2131	172.83%
Gamma(0.9, 0.9) – 1.33% overestimation			
R(5)	5.4000	5.6437	4.51%
R(10)	2.2407	2.3910	6.71%
R(15)	0.7481	0.8138	8.78%
Gamma(1.5, 1.5) – 5.28% overestimation			
R(5)	5.3294	6.4815	21.62%
R(10)	2.-779	2.7679	33.20%
R(15)	0.6138	0.8896	44.93%

Table 2: Net stop-loss premium

The above table suggests that higher stop loss levels x generate weaker net premium bounds. Similarly to the reserve calculations, it also seems that Willmot and Lin's method is required to be very tight (say, < 1% error in the 99.5th percentile) for these net stop-loss premium bounds to be of practical use, even for very small stop-loss levels.

Chapter 4

Conclusions

In investigating Willmot and Lin's compound negative binomial bounds for various loss distributions, several trends appear regardless of the distribution of individual claims. The first was a general tendency that Willmot and Lin's bound improved significantly as ϕ increased. In addition, there were slight improvements in their bound as α decreased.

For specific loss distributions, the exponential case was the only case examined where their bound was sharp and in these cases the use of Willmot and Lin's method is strongly advised due to its speed and sharpness. However, their bound was optimal in one other case; the degenerate loss distribution with $\alpha = 1$, though this situation is virtually non-existent in practice.

The most significant factor in determining the practicality of Willmot and Lin's bound is the "closeness" of the loss distribution (or more generally, its variance) to that of exponential. For distributions such as gamma distributions with $r \approx 1$ and mixed-exponential distributions whose minimum exponential parameter carried most weight, Willmot and Lin's method performed extremely well.

When dealing with fractional weights, only two cases (degenerate and gamma with $r > 1$) generated multiple optimal bounds. However in all cases investigated, the bound generated by the partition with $[\alpha]$ 1's (i.e. $\alpha_j = 1$ for $j = 1,2,\ldots m-1$ and $\alpha_m = \alpha - [\alpha]$) seemed sufficient for practical purposes and fractional values seemed to make only negligible differences in Willmot and Lin's bounds.

Finally for insurance applications, in particular reserve calculations and stop-loss reinsurance, Willmot and Lin's bound were required to be extremely tight (say, < 1% overestimation in the 99.5^{th} percentile or even tighter for higher stop loss levels) for the resulting aggregate distribution to give practical reserve of net stop-loss premium figures. However in both cases, the figures obtained represent strict upper bounds, and can be used in confidence if adopted.

It is possible to extend this investigation and examine even wider varieties of loss distributions or insurance applications. In particular, a natural extension of the mixture of exponentials is to consider a mixture of Erlangs, since all distributions may be decomposed into such a mixture (Willmot and Lin, 2001, §2.1.3).

However the general trends found in this investigation seem to be indicative of more generic trends. The information provided in this investigation should be sufficient to give an individual a reliable guide in determining whether Willmot and Lin's method is practical for their purpose.

Chapter 5

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